A GLOBAL ANALOGUE OF THE SPRINGER RESOLUTION FOR SL_2

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ABSTRACT. The global nilpotent cone $\mathcal{N}$ is a singular stack associated to the choice of an algebraic group G, a smooth projective curve X, and a line bundle $\mathcal{L}$ on X, which is of fundamental importance to the Geometric Langlands Program, and which is of emerging importance to the Classical Langlands Program. In analogy with the ordinary Springer resolution, we construct and study a resolution of singularities of $\mathcal{N}$ in the special case where $G = SL_2$. As an immediate application, we prove that $\mathcal{N}$ is equidimensional and also provide an enumeration of its irreducible components. We hope this is the first step in constructing a global Springer resolution for an arbitrary reductive group.

1. INTRODUCTION

Fix a triple $(X, G, \mathcal{L})$, in which X denotes a smooth projective curve, G denotes a reductive algebraic group, and $\mathcal{L}$ indicates a line bundle on X. Given this data, we may consider the Hitchin moduli stack $\mathcal{M}$, as well as the Hitchin fibration

$$\chi^{\text{Hit}} : \mathcal{M} \to \mathcal{A},$$

where $\mathcal{A}$ denotes a vector space. Beginning with [Hit87a], the Hitchin fibration has received much attention in recent years, playing a significant role in the Geometric Langlands Program ([BD], [Fre07]), the Classical Langlands Program ([Ngô10]), and non-Abelian Hodge theory ([Sim92], [dCHM12]).

While there are various points of view on the Hitchin fibration, we prefer to think of it as a global analogue of the adjoint quotient map associated to the Lie algebra $\mathfrak{g}$ of G. Then, just as the ordinary nilpotent cone $\mathcal{N}$ilp is defined as the fiber over zero of the adjoint quotient map, we may define the global nilpotent cone $\mathcal{N}$ to be the fiber over zero of the Hitchin fibration. While $\mathcal{N}$ilp has played a significant role in geometric representation theory for over 30 years (see Section 1.1), $\mathcal{N}$ has only received a modest, but important, study in the Geometric Langlands Program, and has received almost no study whatsoever in the Classical Langlands Program and in non-Abelian Hodge theory.[1] This relative lack of attention can largely be explained by the fact that $\mathcal{N}$ is the most difficult Hitchin fiber to understand, in part because it is the most singular.

Thus, the purpose of this paper is to begin a program which aims to understand the global nilpotent cone as thoroughly as the ordinary nilpotent cone. To this end, we construct an explicit resolution of singularities of $\mathcal{N}$ when $G = SL_2$, which we think of as a global analogue of the Springer resolution $\widetilde{\mathcal{N}\text{ilp}}$.

More specifically, we identify the Drinfeld/Laumon compactification $\overline{\text{Bun}}_B(X)$ (see Section 2.3) as the appropriate global analogue of the flag variety G/B, and then define the partial global Springer resolution $\widehat{\mathcal{N}}$ (see Definition 3.3) to be a particular closed substack of

$$\mathcal{N} \underset{\text{Bun}_G(X)}{\times} \overline{\text{Bun}}_B(X).$$

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[1] It is common to study the restriction of the Hitchin fibration to various subspaces of $\mathcal{A}$ which, in particular, do not contain zero.

Theorem (see Sections 3.3 and 3.4). *If X is a rational or elliptic curve, then $\widehat{\mathcal{N}}$ is a resolution of singularities of $\mathcal{N}$.*

When the genus of X is greater than 1, $\widehat{\mathcal{N}}$ is not a resolution of singularities of $\mathcal{N}$ because it is not smooth. In order to understand its singularities, we instead study a simpler stack, which is denoted $C\mathcal{G}(X)$, by producing a smooth map

$$\widehat{\mathcal{N}} \to C\mathcal{G}(X).$$

The stack $C\mathcal{G}(X)$ is easier to understand than $\widehat{\mathcal{N}}$ because it is simply the moduli of pairs (λ, s), where λ is a line bundle on X such that $h^0(X, \lambda) \geq 1$, and $s \in H^0(X, \lambda)$. Thus, the geometry of $\widehat{\mathcal{N}}$ is closely tied to the classical geometry of line bundles and divisors on curves, as studied in [ACGH85].

We are then able to define the global Springer resolution $\widetilde{\mathcal{N}}$, which is birationally equivalent to $\widehat{\mathcal{N}}$, by first resolving the singularities of $C\mathcal{G}(X)$.

Theorem (see Section 3.5). *If the genus of X is at least 2, then $\widetilde{\mathcal{N}}$ is a resolution of singularities of $\mathcal{N}$.*

As an immediate consequence, we obtain a proof that $\mathcal{N}$ is equidimensional. While this fact is known when $\mathcal{L}$ is the canonical bundle ω_X (see, for example, [Gin01]), it does not seem to have appeared in print for general $\mathcal{L}$. Furthermore, we are able to give an enumeration of the irreducible components of $\mathcal{N}$

Corollary (see Sections 3.3-3.5). *Suppose that $\deg(\mathcal{L}) \geq 2g$. Then $\mathcal{N}$ is equidimensional of dimension $\deg(\mathcal{L}) + g - 1$. Its irreducible components are indexed by:*

(1) *All integers $d > -\frac{1}{2} \deg(\mathcal{L})$.*
(2) *The square roots of $\mathcal{L}^{-1}$, of which there are 2^{2g}.*

In Section 3.6, a similar analysis is carried out for the the stable part of $\mathcal{N}$, which has only finitely many irreducible components.

1.1. **The ordinary Springer resolution.** Let G be a complex reductive group with Lie algebra $\mathfrak{g}$. Then the adjoint action of G on $\mathfrak{g}$ induces the *adjoint quotient map*

$$\chi : \mathfrak{g} \to \mathfrak{c},$$

where $\mathfrak{c}$ denotes the adjoint quotient $\mathfrak{g}//G$. The notation χ is chosen because when $G = GL_n$, the adjoint quotient map is simply the map which associates to a matrix A the (non-leading coefficients of the) characteristic polynomial of A. Generalizing the fact that the matrix A is nilpotent if and only if its characteristic polynomial is t^n, we define the *nilpotent cone* of $\mathfrak{g}$ to be

$$\mathcal{N}\text{ilp} := \chi^{-1}(0).$$

$\mathcal{N}$ilp is a singular algebraic variety which is normal and which possesses an explicit resolution of singularities known as the *Springer resolution* ([Spr76]). To describe this resolution, let G/B denote the flag variety of G, which may be described as the variety of Borel subalgebras of $\mathfrak{g}$. Letting

$$\widetilde{\mathcal{N}\text{ilp}} := \{(x, \mathfrak{b}) \in \mathcal{N}\text{ilp} \times G/B : x \in \mathfrak{b}\},$$

the Springer resolution is given by the projection map

$$\mu : \widetilde{\mathcal{N}\text{ilp}} \to \mathcal{N}\text{ilp}.$$

Furthermore, there is an isomorphism $\widetilde{\mathcal{N}\text{ilp}} \simeq T^*(G/B)$ between the total space of the Springer resolution and the cotangent bundle of the flag variety. Letting $\widetilde{\mathfrak{g}}$ denote the variety of pairs $(x, \mathfrak{b}) \in \mathfrak{g} \times G/B$ such that $x \in \mathfrak{b}$, we have thus far described the following spaces and maps:

(1.1)

$$
\begin{array}{ccccc}
G/B & \longleftarrow & \widetilde{\mathcal{N}\text{ilp}} \simeq T^*(G/B) & \hookrightarrow & \widetilde{\mathfrak{g}} \\
& & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle p} \\
& & \mathcal{N}\text{ilp} & \hookrightarrow & \mathfrak{g} \\
& & \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \chi} \\
& & \{0\} & \hookrightarrow & \mathfrak{c}
\end{array}
$$

Although this paper is only concerned with geometry, much of the motivation for the Springer resolution comes from representation theory. Given $x \in \mathcal{N}\text{ilp}$, the corresponding fiber in $\widetilde{\mathcal{N}\text{ilp}}$ is known as a *Springer fiber*. Then Springer theory, in its original form, is concerned with the action of the Weyl group W on the top-dimensional cohomology groups of the Springer fibers, as constructed by T.A. Springer in [Spr76]. Afterwards, several equivalent sheaf theoretic versions of the Weyl group action were constructed. We briefly describe three of them, although there are more. Our ultimate goal is to generalize at least one of these constructions to the global setting.

1.1.1. *Springer Theory via perverse sheaves.* The restriction of $p : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ to the regular semisimple locus $\mathfrak{g}^{\text{rs}}$ is the W-torsor denoted $\widetilde{\mathfrak{g}}^{\text{rs}}$. Hence $L := p_*\mathbb{Q}_{\widetilde{\mathfrak{g}}^{\text{rs}}}$ is a W-local system on $\mathfrak{g}^{\text{rs}}$. Furthermore, the smallness of the map p implies that $Rp_*\mathbb{Q}_{\widetilde{\mathfrak{g}}}$ is isomorphic to the intersection cohomology sheaf $IC_{\mathfrak{g}}(L)$ associated to the local system L. Finally, by proper base change, the *Springer sheaf* $R\mu_*\mathbb{Q}_{\widetilde{\mathcal{N}\text{ilp}}}$ is the restriction of $IC_{\mathfrak{g}}(L)$ to $\mathcal{N}\text{ilp}$. The W-action on L is then carried over to the Springer sheaf, which, by the semismallness of μ, is a perverse sheaf which then decomposes into simple components according to the regular representation of W. See [Lus81] and [BM83].

1.1.2. *Springer theory via convolution.* Consider the Steinberg variety $Z := \widetilde{\mathcal{N}\text{ilp}} \underset{\mathcal{N}\text{ilp}}{\times} \widetilde{\mathcal{N}\text{ilp}}$. Then Z is a union of conormal bundles in $G/B \times G/B$ indexed by W, whose closures form the irreducible components of Z. In general (i.e., when μ is replaced by any proper map between algebraic varieties and Z is the corresponding fiber product), there is an algebra isomorphism

$$
H^*(Z, \mathbb{Q}) \simeq \text{Ext}^*(R\mu_*\mathbb{Q}_{\widetilde{\mathcal{N}\text{ilp}}}, R\mu_*\mathbb{Q}_{\widetilde{\mathcal{N}\text{ilp}}}).
$$

However, in the case of the Steinberg variety and Springer resolution, Z is equidimensional of dimension $\dim_{\mathbb{C}}(\mathcal{N}\text{ilp})$, and the middle dimension algebra $H(Z) := H^{\dim_{\mathbb{C}}(\mathcal{N}\text{ilp})}(Z, \mathbb{Q})$ is a subalgebra of $H^*(Z, \mathbb{Q})$ which induces the following two algebra isomorphisms:

$$
\mathbb{Q}[W] \simeq H(Z) \simeq \text{End}(R\mu_*\mathbb{Q}_{\widetilde{\mathcal{N}\text{ilp}}}, R\mu_*\mathbb{Q}_{\widetilde{\mathcal{N}\text{ilp}}}).
$$

See [CG97, Ch. 3].

1.1.3. *Springer theory via nearby cycles.* Let ψ_χ denote the nearby cycles functor associated to the adjoint quotient map. Then ψ_χ is a functor from (the derived categories of) constructible sheaves on $\mathfrak{g}$ to constructible sheaves on $\mathcal{N}\text{ilp}$. There is then an isomorphism

$$
\psi_\chi(\mathbb{Q}_{\mathfrak{g}}) \simeq R\mu_*\mathbb{Q}_{\widetilde{\mathcal{N}\text{ilp}}}.
$$

Furthermore, letting U^{rs} denote the intersection of a small ball around the origin in $\mathfrak{c}$ with the regular semisimple part of $\mathfrak{c}$, there is an action of $\pi_1(U^{\mathrm{rs}})$ on $\psi_\chi(\mathbb{Q}_\mathfrak{g})$. Although $\pi_1(U^{\mathrm{rs}})$ is isomorphic to the Braid group of W, the action on the Springer sheaf factors through W. See [Gri98, Sec. 2.2], [Slo80], and [Hot81].

1.2. **What is meant by a global analogue?** Returning to diagram (1.1), a closer examination reveals that all maps appearing in (1.1) are G-equivariant and that all spaces appearing in (1.1) are only considered up to conjugation by G. We may thus assert that ordinary Springer theory is in fact about the study of the $\mathbb{C}$-points of the associated quotient stacks of (1.1):

(1.2)
$$
\begin{array}{ccc}
G/B \longleftarrow (\widetilde{\mathcal{N}\mathrm{ilp}}/G)(\mathbb{C}) \hookrightarrow (\widetilde{\mathfrak{g}}/G)(\mathbb{C}) \\
\downarrow \qquad\qquad\qquad \downarrow \\
(\mathcal{N}\mathrm{ilp}/G)(\mathbb{C}) \hookrightarrow (\mathfrak{g}/G)(\mathbb{C}) \\
\downarrow \qquad\qquad\qquad \downarrow \\
\{0\} \hookrightarrow \mathfrak{c}
\end{array}
$$

The use of the adjective 'global' may then be roughly described as an attempt to create a theory in which the point $\mathrm{Spec}(\mathbb{C})$ is replaced by a complex projective curve X which is smooth and connected. Although it will not suffice to literally replace $\mathrm{Spec}(\mathbb{C})$ with X, we will still take seriously the idea of creating "a family of Springer theories indexed by the curve X."

Luckily, global analogues of the Lie algebra $\mathfrak{g}$, the adjoint quotient $\mathfrak{c}$, and the adjoint quotient map χ have already received significant study and attention in the form of the *Hitchin fibration* ([Hit87a])

$$\chi^{\mathrm{Hit}} : \mathcal{M} \to \mathcal{A}.$$

$\mathcal{M}$ denotes the *Hitchin moduli stack* parameterizing Higgs bundles on X, and $\mathcal{A}$ denotes the *Hitchin base space*. Although we have not indicated so in the notation, the definition of $\mathcal{M}$ depends on choosing the data of a smooth projective curve X, a reductive group G, and a line bundle[2] $\mathcal{L}$ on X. When $\mathcal{L}$ is the canonical bundle of X, we have an identification

$$\mathcal{M} \simeq T^* \mathrm{Bun}_G(X).$$

We refer the reader to Section 2.2 for definitions and for a justification of why the Hitchin fibration is a global analogue of $\chi : \mathfrak{g} \to \mathfrak{c}$.

In analogy with ordinary Springer theory, we define the *global nilpotent cone*

$$\mathcal{N} := (\chi^{\mathrm{Hit}})^{-1}(0).$$

Having now identified global analogues for $\mathcal{N}\mathrm{ilp}, \mathfrak{g}, \mathfrak{c}$, and χ, the obvious goal would then be to find a resolution of singularities of $\mathcal{N}$, providing a global analogue of the entirety of (1.2). This is precisely what we will do in this paper in the particular case of $G = SL_2$. We hope to be able to extend our construction to SL_n (and possibly arbitrary reductive G) in a future paper.

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[2]In fact, the line bundle is fixed to be the canonical bundle in the original Hitchin moduli space ([Hit87a]).

1.2.1. *Relation to Yun's global Springer theory.* Recently, Z. Yun has extensively developed a global Springer theory in [Yun11]. Rather than viewing the Hitchin fibration as a global analogue of the adjoint quotient map, Yun views it as a global analogue of the map in *local Springer theory* which is most similar to $\widetilde{\mathfrak{g}} \to \mathfrak{g}$ in ordinary Springer theory[3]. While this paper is primarily concerned with the geometry of the global nilpotent cone, Yun's global Springer theory is largely concerned with extending the representation theoretic study of ordinary Springer theory to the global setting. In particular, Yun constructs the *parabolic Hitchin moduli stack* $\mathcal{M}^{\mathrm{par}}$, which, besides classifying Higgs bundles (E, φ), also adds the additional data of a point $x \in X$ and a choice of B-reduction of the fiber E_x which is compatible with φ. Then the *parabolic Hitchin fibration*

$$\chi^{\mathrm{par}} : \mathcal{M}^{\mathrm{par}} \to \mathcal{A} \times X$$

assumes the role of $\widetilde{\mathfrak{g}} \to \mathfrak{g}$, and an action of the affine Weyl group is defined on $R\chi^{\mathrm{par}}_* \overline{\mathbb{Q}}_\ell$. Although we have claimed that both the Hitchin fibration and parabolic Hitchin fibration are analogues of $\widetilde{\mathfrak{g}} \to \mathfrak{g}$, the difference essentially comes from differing notions of affine (i.e., local) Springer fiber ([Lus96]). Under this analogy, Hitchin fibers correspond to affine Springer fibers which live in the affine Grassmannian, while parabolic Hitchin fibers correspond to affine Springer fibers living in the affine flag variety.

We emphasize, though, that $\mathcal{M}^{\mathrm{par}}$ is only studied over a subspace of the Hitchin base, which in characteristic 0 coincides with the anisotropic subspace of $\mathcal{A}$, that *does not include the zero section.* Therefore the global nilpotent cone plays no role in Yun's global Springer theory.

1.3. **Further motivation.** We mention here two other sources of motivation for the study of $\mathcal{N}$.

1.3.1. *Geometric Langlands.* So far, one of the few results regarding the global nilpotent cone is a theorem stating that $\mathcal{N}$ is a Lagrangian substack of $T^* \mathrm{Bun}_G(X)$ ([Lau88],[Gin01],[Fal93]). This has several important consequences, as proven in [BD].

(1) The Hitchin fibration is flat of relative dimension $\dim(\mathcal{N}) = \dim(\mathrm{Bun}_G(X))$.
(2) The stack $\mathrm{Bun}_G(X)$ is *good.* In particular, this means that $T^* \mathrm{Bun}_G(X)$, which is a priori a derived algebraic stack, is in fact an ordinary algebraic stack.
(3) Any D-module on $\mathrm{Bun}_G(X)$ whose singular support is contained in $\mathcal{N}$ is holonomic.

It is the third point that is especially relevant to the Geometric Langlands Program. Given a smooth projective curve X along with reductive group G and its Langlands dual $G^\vee$, an important problem in the Geometric Langlands Program is whether one can associate to any $G^\vee$-local system $\mathcal{F}$, a D-module $M_\mathcal{F}$ on $\mathrm{Bun}_G(X)$, known as the *Hecke eigensheaf,* which has eigenvalue $\mathcal{F}$. While this problem is still open in general, A. Beilinson and V. Drinfeld have defined a substack of $\mathrm{LocSys}_{G^\vee}(X)$, the stack of $G^\vee$-local systems on X, parameterizing so-called $G^\vee$-opers, and for which they were able to construct corresponding Hecke eigensheaves. The singular support of these Hecke eigensheaves is the global nilpotent cone $\mathcal{N}$ ([BD]).

1.3.2. *Fundamental Lemma.* The Fundamental Lemma, whose proof was recently completed B.C. Ngô in [Ngô10] (see also the survey [Nad12]), is an identity relating the κ-orbital integral of an anisotropic element γ_G of a reductive group G defined over a number field F with the stable orbital integral of the transferred element γ_H, which lives in an endoscopic subgroup H of G. Ngô reduces the proof to the study of the cohomology of Hitchin fibers. Similarly, there is also a variant known as the Weighted Fundamental Lemma, whose proof was completed by P-H. Chaudouard and G. Laumon ([CL10]) using methods similar to those of Ngô. However, in both cases, it is not necessary

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[3]There is precedent for this point of view in [Ngô10], in which Hitchin fibers prove to be more manageable than affine Springer fibers.

to understand all Hitchin fibers, but only those living over the anisotropic[4] and generically regular semi-simple subspaces, respectively.

Since 0 is not an element of the anisotropic or generically regular semisimple loci, neither the proof of the Fundamental Lemma nor of the Weighted Fundamental Lemma takes the global nilpotent cone into consideration. As communicated to the author by Ngô, it is nevertheless likely that the global nilpotent cone fits into the theory of orbital integrals. Indeed, orbital integrals associated to a regular semisimple element living in a sufficiently small neighborhood of the identity possess an asymptotic expansion in terms of orbital integrals over *unipotent* conjugacy classes, whose coefficients are known as *Shalika germs* ([Sha72]). According to Ngô, there should be a close relationship between these Shalika germs and the global nilpotent cone.

1.4. **Contents of the paper.**

There are two main sections to this paper. In the first section, we provide the background necessary to construct a global Springer resolution for SL_2. This starts with a brief review of the ordinary Springer resolution, which motivates everything that follows. Global analogues of the nilpotent cone and the flag variety are then introduced. In the former case, this leads us to a discussion of the Hitchin fibration and global nilpotent cone. In the latter case, we review the Drinfeld and Laumon compactifications of the stack of B-bundles on X. Finally, we end by reviewing the geometry of line bundles and linear systems on X. This includes a definition of the stack $C\mathcal{G}(X)$, which later plays a crucial role in understanding and constructing a resolution of singularities of the global nilpotent cone.

The other main section of the paper revolves around the construction of a global Springer resolution, along with the study of its geometric properties. As an intermediate step, we first define the *partial global Springer resolution* $\widehat{\mathcal{N}}$. We then verify that the projection map from $\widehat{\mathcal{N}}$ to the global nilpotent cone is a proper, birational equivalence, which is furthermore a resolution of singularities when the genus of X is at most one. In general, we are able to understand the singularities of $\widehat{\mathcal{N}}$ by producing a smooth map to $C\mathcal{G}(X)$ and studying its singularities instead. By pulling back a resolution of singularities of $C\mathcal{G}(X)$ to $\widehat{\mathcal{N}}$, we are finally able to define the *global Springer resolution* $\widetilde{\mathcal{N}}$. This allows us to obtain corollaries on the equidimensionality of the global nilpotent cone along with an enumeration of its irreducible components. We end the paper with a discussion of the stable part of the global nilpotent cone, as well as a description of $\widetilde{\mathcal{N}}$ in the case where the twisting line bundle is of low degree.

1.5. **Conventions and notation.**

1.5.1. *Conventions.* We work over the complex numbers $\mathbb{C}$. All geometric objects (schemes, stacks) discussed will be defined over $\mathbb{C}$. Similarly, all algebraic groups will also be defined over $\mathbb{C}$.

We will work heavily throughout this paper with algebraic stacks. We will think of an algebraic stack Y as functor

$$Y : \text{Schemes}/\mathbb{C} \to \text{Groupoids}$$

from the category of schemes over $\mathbb{C}$ to the category of groupoids. Given a scheme S, the groupoid of S-points of Y will be denoted $Y(S)$. We will typically only describe the objects of $Y(S)$, as it should be clear what the morphisms are. Given a commutative $\mathbb{C}$-algebra R, we will also sometimes discuss the R-points of Y, by which we will mean the $\text{Spec}(R)$-points of Y. For the types of stacks discussed in this paper, we recommend [Hei] for a concise introduction.

Although we will typically discuss the S-points of Y, we note that it actually suffices to only consider R-points, using the fact that Y is a sheaf and the fact that any scheme is a limit of affine schemes. Furthermore, since any ring is a colimit of finitely generated rings, we may assume that

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[4]For technical reasons, it is actually necessary in positive characteristic to work with a slightly smaller subspace.

S is a scheme of finite type over $\mathbb{C}$. This fact will be useful in allowing us to use scheme theoretic results with finite type hypotheses, which we will do implicitly without further mention.

Lastly, we note that there is no difference between the geometric points of S and the closed points of S, as S is defined over an algebraically closed field. When stating definitions and results that apply more generally to geometric points of S, we will refer to them as geometric points. Likewise for closed points.

1.5.2. *Notation.* X will denote a smooth, projective, connected algebraic curve of genus g, and $\mathcal{L}$ will denote a fixed line bundle on X. In Section 2, there will be no assumptions placed on the degree of $\mathcal{L}$, while in Section 3 we will assume that the degree of $\mathcal{L}$ is even and no less than $2g$ (with the exception of Section 3.6).

Since we will make several infinitesimal arguments throughout the paper, we will use the notation $S[\epsilon]$ (resp., $R[\epsilon]$) to denote the product of the scheme S with the dual numbers (resp., the tensor product of the ring R with the ring of dual numbers).

As the S-points of the stacks we work with will typically parameterize S-families of objects on X, the product $X \times S$ will be conveniently denoted by X_S. It is equipped with projection maps

$$
\begin{array}{ccc}
X_S & \xrightarrow{\ \pi\ } & X \\
\ \downarrow{\scriptstyle p} & & \\
S & &
\end{array}
$$

Most of the objects we work with in this paper will be vector bundles, and more generally coherent sheaves, on X_S. Given a point $x \in X_S$ and a coherent sheaf $\mathcal{F}$ on X_S, we will let $\mathcal{F}_x$ denote the *fiber* of $\mathcal{F}$ at x (as opposed to the stalk). We use instead the notation $i_x^* \mathcal{F}$ to denote the stalk of $\mathcal{F}$ at x. We will only have occasion to discuss stalks in the proof of Proposition 3.2.

We let G denote a complex reductive algebraic group whose derived subgroup $[G, G]$ is simply connected. The group B denotes a Borel subgroup of G, and T denotes a maximal torus of G, which will frequently be assumed to be $B/[B, B]$. For any group H, we use the notation $\mathrm{Bun}_H(X)$ for the stack of principal H-bundles on X. When dealing with principal H-bundles, recall that there is a vector bundle associated to any choice of an H-bundle and a representation of H. Given an H-bundle E and a representation V of H, the associated vector bundle $E \overset{H}{\times} V$ is denoted by V_E. In the special case of the adjoint action of H on its Lie algebra $\mathfrak{h}$, we let $\mathrm{ad}(E) := E \overset{H}{\times} \mathfrak{h}$.

Finally, we will frequently be in the situation of a group H acting on a space Y. We will use the notation Y/H for the stack-theoretic quotient, and the notation $Y/\!/H$ for the GIT quotient.

2. Background

We begin this section with a brief review of the ordinary Springer resolution of the nilpotent cone $\mathcal{N}\mathrm{ilp} \subset \mathfrak{g}$. We then review basic facts and definitions regarding the Hitchin fibration and the global nilpotent cone $\mathcal{N}$. After this we review the stack $\mathrm{Bun}_B(X)$ and its two relative compactifications over $\mathrm{Bun}_G(X)$, due to Drinfeld and Laumon. Finally, we end the section by reviewing some classical geometry related to varieties of line bundles and linear systems on curves, as well as their stack-theoretic counterparts.

2.1. **The Springer resolution.** We begin by giving the definition of the nilpotent cone $\mathcal{N}\mathrm{ilp} \subset \mathfrak{g}$. In order to do so in a way that generalizes nicely to the global situation, we must first recall the adjoint quotient map.

Definition 2.1. We call $\mathfrak{c} := \mathfrak{g}//G$ the *adjoint quotient* of $\mathfrak{g}$ and the corresponding map $\chi : \mathfrak{g} \to \mathfrak{c}$ the *adjoint quotient map*.

Recall that when $\mathfrak{g} = \mathfrak{gl}_n$, the adjoint quotient map may be viewed as the map sending a matrix A to the non-leading term coefficients of its characteristic polynomial. Therefore $A^n = 0$ if and only if $\chi(A) = 0$. This leads to the definition of the nilpotent cone.

Definition 2.2. $\mathcal{N}\mathrm{ilp} := \chi^{-1}(0)$ is called the *nilpotent cone* of $\mathfrak{g}$.

In order to properly make sense of the definition of $\mathcal{N}\mathrm{ilp}$, we recall a result of Chevalley which states that $\mathfrak{c} \simeq \mathbb{A}^r$, where r denotes the rank of $\mathfrak{g}$ ([Bou68]). More precisely, this isomorphism is realized by the existence of r independent generators of the ring of functions of $\mathfrak{c}$. The generators are each homogeneous of degrees $d_1, \ldots, d_r$. Letting e_i denote the ith exponent of $\mathfrak{g}$, we have the relation

$$d_i = e_i + 1.$$

It is straightforward to see from the definition that $\mathcal{N}$ilp is a singular affine variety (for example, the origin is always singular). The singularities of $\mathcal{N}$ilp have been completely classified in Lie theoretic terms, as follows.

Definition 2.3. An element $x \in \mathfrak{g}$ is said to be *regular* if its centralizer $Z_{\mathfrak{g}}(x)$ has the minimal possible dimension r $(= \text{rk}(\mathfrak{g}))$. We let $\mathfrak{g}^{\text{reg}} \subset \mathfrak{g}$ denote the locus of regular elements.

Proposition 2.4. *The smooth locus of $\mathcal{N}$ilp is given by the regular elements of $\mathcal{N}$ilp. Furthermore, $\mathcal{N}$ilp is a normal algebraic variety.*

Proof. See [HTT08, Sec. 10.3]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

In order to give some motivation for constructing a resolution of singularities of $\mathcal{N}$ilp using Proposition 2.4, we recall that for a regular nilpotent $x \in \mathfrak{g}$, there is a unique Borel subalgebra containing x. If we identify the flag variety G/B with the variety of Borel subalgebras of $\mathfrak{g}$, we may define

$$\widetilde{\mathcal{N}\text{ilp}} := \{(x, \mathfrak{b}) \in \mathcal{N}\text{ilp} \times G/B : x \in \mathfrak{b}\}.$$

The projection map $\widetilde{\mathcal{N}\text{ilp}} \to \mathcal{N}$ilp is proper (since G/B is compact) and is a birational equivalence (inducing an isomorphism over the regular part of $\mathcal{N}ilp$). Finally, $\widetilde{\mathcal{N}\text{ilp}}$ is smooth because it is a vector bundle over G/B.

Theorem 2.5. $\widetilde{\mathcal{N}\text{ilp}} \to \mathcal{N}$ilp *is a resolution of singularities. Furthermore, $\widetilde{\mathcal{N}\text{ilp}} \simeq T^*(G/B)$.*

Proof. See [CG97, Sec. 3.2]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

The resolution of singularities $\widetilde{\mathcal{N}\text{ilp}} \to \mathcal{N}$ilp is known as the *Springer resolution.*

Remark 2.6. When $G = SL_n$, we wish to emphasize an alternative but equivalent formulation of $\widetilde{\mathcal{N}\text{ilp}}$. Rather than identifying $\text{Fl}_n := SL_n/B$ with the variety of Borel subalgebras, we may instead identify it with the variety of flags $V_1 \subset V_2 \subset \ldots \subset V_n$ in which V_i is a vector space of dimension i. It is then straightforward to check that the variety of pairs $(A, (V_1 \subset \ldots \subset V_n)) \in \mathcal{N}\text{ilp} \times \text{Fl}_n$ such that $A \cdot V_i \subset V_i$ for all i is the same as $\widetilde{\mathcal{N}\text{ilp}}$. Furthermore, the condition that $A \cdot V_i \subset V_i$ is equivalent to $A \cdot V_i \subset V_{i-1}$ since A is nilpotent.

2.2. **The Hitchin fibration and global nilpotent cone.** In introducing the Hitchin fibration and global nilpotent cone, we would like to emphasize their analogy with finite dimensional Lie algebras and classical Springer theory as reviewed in Section 2.1. As discussed in Section 1.2, we may obtain a first approximation to what we will call 'global Springer theory' by replacing the point $\text{Spec}(\mathbb{C})$ by the curve X in (1.2). We begin by defining the global analogue of the Lie algebra $\mathfrak{g}$.

Definition 2.7. Given the group G and curve X, together with the additional data of a line bundle $\mathcal{L}$ on X, we define, following [Ngô10], the *Hitchin moduli stack* $\mathcal{M}_{X,G,\mathcal{L}}$ to be the mapping stack $\text{Hom}(X, \mathfrak{g}_{\mathcal{L}}/G)$. Recall that $\mathfrak{g}_{\mathcal{L}}/G$ denotes the stack quotient and that $\mathfrak{g}_{\mathcal{L}}$ denotes the associated vector bundle given by viewing $\mathfrak{g}$ as a representation of $\mathbb{G}_m$. We will typically denote $\mathcal{M}_{X,G,\mathcal{L}}$ by $\mathcal{M}$ if the triple $(X, G, \mathcal{L})$ is clear from the context.

Remark 2.8. Note that, after a choice of trivialization, we can roughly think of a $\mathbb{C}$-point of $\mathcal{M}$ as a collection of elements $\varphi_x \in \mathfrak{g}$ for every point $x \in X$.

While Definition 2.7 is conceptually useful for the transition to a global Springer theory, the following lemma provides a more standard definition of the Hitchin moduli stack.

Lemma 2.9. $\mathcal{M}$ *is equivalent to the stack whose S-points consist of all pairs (E, φ) where*

    (1) *E is a principal G-bundle on X_S and*

(2) $\varphi \in H^0(X_S, \mathrm{ad}(E) \otimes \pi^* \mathcal{L})$.

Proof. The proof is essentially tautological. Recall that an S-point of $\mathrm{Hom}(X, \mathfrak{g}_{\mathcal{L}}/G)$ is the same as an X_S point of $\mathfrak{g}_{\mathcal{L}}/G$. $\square$

The pair (E, φ) appearing in Lemma 2.9 will be referred to as a *Higgs bundle*, and the section φ will be referred to as a *Higgs field*.

Example 2.10. Let us give an equivalent formulation of a Higgs bundle when $G = SL_n$ which will be useful in subsequent sections. A principal SL_n-bundle is equivalent to a rank n vector bundle E together with an isomorphism $\det(E) \simeq \mathcal{O}_X$. A Higgs field is then equivalent to giving a twisted endomorphism $\varphi : E \to E \otimes \mathcal{L}$ such that $\mathrm{tr}(\varphi) = 0$. We will conflate these equivalent notions of Higgs bundle whenever $G = SL_n$.

Having found a suitably global version of the Lie algebra $\mathfrak{g}$, we now formulate a global version of the adjoint quotient $\mathfrak{c}$ and the adjoint quotient map χ.

Definition 2.11. Define the *Hitchin base space* $\mathcal{A}_{X,G,\mathcal{L}}$ (or simply $\mathcal{A}$ if the context is clear) to be the space of global sections of the affine bundle $\mathfrak{c}_{\mathcal{L}} := \mathcal{L} \overset{\mathbb{G}_m}{\times} \mathfrak{c}$ on X.

The following lemma gives a more concrete (but less canonical) description of $\mathcal{A}$. This description of $\mathcal{A}$ can be originally found in [Hit87b].

Lemma 2.12. *Recalling the notation from Section 1.1, there is a non-canonical isomorphism*

$$(2.1) \qquad \mathcal{A} \simeq \bigoplus_{i=1}^{r} H^0(X, \mathcal{L}^{\otimes d_i}).$$

Proof. Choosing generators $f_1, \ldots, f_r$ of degrees $d_1, \ldots, d_r$ for the ring of functions of $\mathfrak{c}$ determines an action of $\mathbb{G}_m$ on $\mathfrak{c}$, from which it is easily checked that $\mathfrak{c}_{\mathcal{L}} \simeq \bigoplus_{i=1}^{r} \mathcal{L}^{\otimes d_i}$. This gives the isomorphism of (2.1), which is non-canonical due to the choice of generators $f_1, \ldots, f_r$. $\square$

In order to construct a global analogue of the map χ, notice that χ is both $\mathbb{G}_m$-equivariant and G-invariant. Therefore χ induces a map $\mathfrak{g}_{\mathcal{L}}/G \to \mathfrak{c}_{\mathcal{L}}$, from which we obtain a map

$$\chi^{\mathrm{Hit}} : \mathcal{M} \to \mathcal{A}$$

known as the *Hitchin fibration*.

Remark 2.13. Following up on the informal commentary of Remark 2.8, the map χ^{Hit} may be roughly thought of as associating to a Higgs bundle (E, φ) the collection of elements $\chi(\varphi_x) \in \mathfrak{c}$ for every point $x \in X$.

Example 2.14. In the particular case of $G = SL_n$, the degrees of the generators of $\mathfrak{c}$ are given by $2, \ldots, n$, which are exactly the degrees of the coefficients of the characteristic polynomial of an element $A \in \mathfrak{sl}_n$, viewed as symmetric polynomials in the eigenvalues of A. Then, as alluded to above, $\chi^{\mathrm{Hit}} : \mathcal{M} \to \mathcal{A}$ can be thought of as a global characteristic polynomial map. Indeed, for any point $x \in X$, the fiber $\chi^{\mathrm{Hit}}(E, \varphi)_x$ is nothing but the characteristic polynomial of φ_x. The absence of $H^0(X, \mathcal{L})$ appearing as a direct summand of $\mathcal{A}$ comes from the condition that $\mathrm{tr}(\varphi) = 0$.

Having formulated a global analogue of χ, we may now define the global nilpotent cone, first introduced in [Lau87] and [Lau88].

Definition 2.15. The *global nilpotent cone* $\mathcal{N}_{X,G,\mathcal{L}}$ (typically just written as $\mathcal{N}$) is defined to be the reduced substack of $(\chi^{\mathrm{Hit}})^{-1}(0)$, the fiber over zero of the Hitchin fibration. It is therefore a closed substack of the Hitchin moduli stack. A pair $(E, \varphi) \in \mathcal{N}$ will be referred to as a nilpotent Higgs bundle, and φ will be called a nilpotent Higgs field.

Example 2.16. When $G = SL_n$, a Higgs bundle (E, φ) is nilpotent if and only if $\varphi^n = 0$, where $\varphi^n = (\phi \otimes \mathrm{id}_{\mathcal{L}^{\otimes n-1}}) \circ \ldots \circ (\varphi \otimes \mathrm{id}_{\mathcal{L}}) \circ \varphi$ is a map $E \to E \otimes \mathcal{L}^{\otimes n}$.

Lastly, it will be useful to generalize the notion of a regular element of $\mathfrak{g}$ to the global setting.

Definition 2.17. The *regular* locus of the Hitchin moduli stack is the substack of $\mathcal{M}$ defined to be $\mathcal{M}^{\mathrm{reg}} := \mathrm{Hom}(X, \mathfrak{g}_{\mathcal{L}}^{\mathrm{reg}}/G)$. A Higgs bundle $(E, \varphi) \in \mathcal{M}(S)$ will be called a *regular Higgs bundle* (or just *regular*) if $(E, \varphi) \in \mathcal{M}^{\mathrm{reg}}(S)$. Furthermore, if $(E, \varphi) \in \mathcal{M}(S)$ corresponds to a map $h : X_S \to \mathfrak{g}_{\mathcal{L}}/G$, we will say that (E, φ) is *generically regular* if there exists an open set $U \subset X$ such that the restriction of h to U_S maps to $\mathfrak{g}_{\mathcal{L}}^{\mathrm{reg}}/G$.

2.3. **A relative compactification of** $\mathrm{Bun}_B(X)$. The goal of this section is to formulate an appropriate global analogue of the flag variety G/B. Let us begin by returning to diagram (1.1) and considering the space of maps $\mathrm{Hom}(X, G/B)$. In order to give an explicit description of $\mathrm{Hom}(X, G/B)$, we will use the Plücker embedding

$$G/B \hookrightarrow \prod_{i=1}^{r} \mathbb{P}(V_{\omega_i}^*),$$

where V_{ω_i} is the fundamental representation of G associated to the fundamental weight ω_i. Since a map $X \to \mathbb{P}^n$ of degree d is the same as specifying a line bundle λ of degree $-d$ with an embedding $0 \to \lambda \to \mathcal{O}_X^{n+1}$ such that the quotient is locally free, we obtain the following Plücker description of $\mathrm{Hom}(X, G/B)$.

Giving a map $X \to G/B$ is equivalent to giving a collection of line *subbundles*

(2.2)
$$\{\lambda_\mu \hookrightarrow \mathcal{O}_X \otimes V_\mu\}$$

for every dominant weight μ satisfying the Plücker relations[5]. A consequence of the Plücker relations is that it suffices to specify line subbundles only for the finitely many fundamental weights of G.

On the other hand, when $G = SL_n$, there is an alternative, flag-like description of $\mathrm{Hom}(X, \mathrm{Fl}_n)$. In this case, giving a map $X \to \mathrm{Fl}_n$ is equivalent to giving a flag of *subbundles*

$$V_1 \subset \ldots \subset V_{n-1} \subset \mathcal{O}_X^n$$

in which $\mathrm{rk}(V_i) = i$.

Even though the flag variety is complete, the space of maps $\mathrm{Hom}(X, G/B)$ is not. In order to correct this defect, there is a compactification of $\mathrm{Hom}(X, G/B)$ due to Drinfeld known as the space of *quasi-maps* ([Kuz97],[Bra06]). This space is obtained by taking the Plücker description of $\mathrm{Hom}(X, G/B)$ in (2.2) and requiring only that the λ_μ be *subsheaves* of $\mathcal{O}_X \otimes V_\mu$. This means that the cokernel of $\lambda_\mu \hookrightarrow \mathcal{O}_X \otimes V_\mu$ may have torsion.

In a similar fashion, when $G = SL_n$, there is a compactification of $\mathrm{Hom}(X, \mathrm{Fl}_n)$ due to Laumon given by considering flags of *subsheaves* $V_1 \subset \ldots \subset V_{n-1} \subset \mathcal{O}_X^n$. The resulting space is known as the space of *quasi-flags,* which coincides with the space of quasi-maps if and only if $n = 2$ ([Kuz97], [Bra06]). A. Kuznetsov has shown that the space of quasi-flags is a small resolution of singularities of the space of quasi-maps in [Kuz97].

Example 2.18. Let us examine the compactification of $\mathrm{Hom}(\mathbb{P}^1, \mathrm{Fl}_2)$, in which the Drinfeld and Laumon compactifications coincide. A degree one map $\mathbb{P}^1 \to \mathrm{Fl}_2$ is equivalent to giving a subbundle

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \oplus \mathcal{O}.$$

Writing z, w for the homogeneous coordinates of $\mathbb{P}^1$, such a map may be written as

$$\begin{pmatrix} az + bw \\ cz + dw \end{pmatrix}, \text{ with } a, b, c, d \in \mathbb{C}.$$

---

[5]We omit the details of the Plücker relations. The interested reader may consult [Kuz97].

The condition that $\mathcal{O}(-1)$ be a subbundle of $\mathcal{O} \oplus \mathcal{O}$ is equivalent to requiring that

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0.$$

Therefore the space of maps $\mathrm{Hom}(\mathbb{P}^1, \mathrm{Fl}_2)$ is isomorphic to the complement of the Plücker embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ inside $\mathbb{P}^3$. The corresponding space of quasi-maps/quasi-flags is simply

$$\mathbb{P} \mathrm{Hom}(\mathcal{O}(-1), \mathcal{O}^2) \simeq \mathbb{P}^3.$$

For the purposes of finding a global Springer resolution, $\mathrm{Hom}(X, G/B)$ and its compactifications will be insufficient. The problem is that in the Plücker (and flag-like, when $G = SL_n$) description of $\mathrm{Hom}(X, G/B)$, all line bundles are subbundles of a trivial vector bundle. The Hitchin moduli stack, on the other hand, lives over the entire moduli of G-bundles. The solution to the problem is to place $\mathrm{Hom}(X, G/B)$ and its compactifications into the larger context of principal B-bundles.

To begin, there is a map of stacks

$$q : \mathrm{Bun}_B(X) \to \mathrm{Bun}_G(X)$$

arising from the inclusion $B \subset G$. It is then straightforward to check that the fiber over the trivial G-bundle E_G^0 is exactly $\mathrm{Hom}(X, G/B)$. Then, just as $\mathrm{Hom}(X, G/B)$ is not complete, the map q is not proper. We would therefore like to have a relative compactification of $\mathrm{Bun}_B(X)$ so that the corresponding fiber over E_G^0 coincides with one of the compactifications of $\mathrm{Hom}(X, G/B)$.

Relative compactifications generalizing quasi-maps and quasi-flags exist, and are known as the Drinfeld and Laumon compactifications, respectively ([BG02]). The Drinfeld compactification, denoted $\mathrm{Bun}_B^D(X)$, exists for any reductive algebraic group G, and is a generalization of the space of quasi-maps. The Laumon compactification, denoted $\mathrm{Bun}_B^L(X)$, only exists when $G = SL_n$, and is a generalization of the space of quasi-flags.

Definition 2.19. The *Drinfeld compactification* $\mathrm{Bun}_B^D(X)$ of $\mathrm{Bun}_B(X)$ is the algebraic stack whose S-points are given by pairs $(\mathcal{F}_T, \mathcal{F}_G)$, where $\mathcal{F}_T$ is a T-bundle and $\mathcal{F}_G$ is a G-bundle. Furthermore, we require that for every dominant weight μ, there is an embedding of coherent sheaves

$$L_{\mathcal{F}_T}^\mu \hookrightarrow V_{\mathcal{F}_G}^\mu.$$

The collection of embeddings for every dominant weight μ is required to satisfy the Plücker relations. See [BG02] for a full description.

Definition 2.20. The *Laumon compactification* $\mathrm{Bun}_B^L(X)$ of $\mathrm{Bun}_B(X)$ is the algebraic stack whose S-points consist of flags of coherent sheaves $V_1 \subset V_2 \subset \ldots \subset V_n$ on X_S such that:

- V_i is a vector bundle of rank i for $1 \leq i \leq n-1$ and V_n is an SL_n-bundle.
- Each quotient V_i/V_{i-1} is S-flat.

Remark 2.21. The difference between an S-point of $\mathrm{Bun}_B^L(X)$ and an S-point of $\mathrm{Bun}_B(X)$ is that an S-point of $\mathrm{Bun}_B(X)$ consists of a flag of vector bundles as in Definition 2.20 such that each quotient V_i/V_{i-1} is X_S-flat. This is equivalent to saying that each V_{i-1} is a subbundle of V_i. Note that when $S = \mathrm{Spec}(\mathbb{C})$, each quotient is automatically S-flat, so each V_{i-1} is simply a coherent subsheaf of V_i.

The following proposition shows that the compactifications $\mathrm{Bun}_B^D(X)$ and $\mathrm{Bun}_B^L(X)$ each have desirable properties.

Proposition 2.22. *The following properties hold for $\mathrm{Bun}_B^D(X)$ (for any reductive algebraic group G) and for $\mathrm{Bun}_B^L(X)$ (when $G = SL_n$).*

(1) *There is a natural inclusion map $j^D : \mathrm{Bun}_B(X) \to \mathrm{Bun}_B^D(X)$ (resp., $j^L : \mathrm{Bun}_B(X) \to \mathrm{Bun}_B^L(X)$) making $\mathrm{Bun}_B(X)$ an open, dense substack.*

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(2) *There is a proper map $q^D : \operatorname{Bun}_B^D(X) \to \operatorname{Bun}_G(X)$ (resp., $q^L : \operatorname{Bun}_B^L(X) \to \operatorname{Bun}_G(X)$) such that $q^D \circ j^D = q$ (resp., $q^L \circ j^L = q$).*

Proof. The density statement may be found in [BG02, Prop 1.2.3]. We remark that the simply-connectedness assumption on $[G, G]$ is necessary here. A proof that q^D is proper may be found in [BG02, Prop 1.2.2]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

For later use, we now record some basic information about $\operatorname{Bun}_B(X)$, and hence its compactifications by the density statement of Proposition 2.22. Letting $T := B/[B, B]$ be the maximal torus of G, there is an induced map

$$r : \operatorname{Bun}_B(X) \to \operatorname{Bun}_T(X),$$

which may be extended to maps

$$r^D : \operatorname{Bun}_B^D(X) \to \operatorname{Bun}_T(X),$$
$$r^L : \operatorname{Bun}_B^L(X) \to \operatorname{Bun}_T(X).$$

The map r^D is the obvious projection map. The map r^L is given by associating to a flag $V_1 \subset \ldots \subset V_n$ the $(n-1)$-tuple of line bundles

$$(V_1, \det(V_2), \ldots, \det(V_{n-1})).$$

Each of these maps induce bijections on connected components. Let us therefore review why the connected components of $\operatorname{Bun}_T(X)$ are in bijection with the coweight lattice Λ_G of G. Given a T-bundle $\mathcal{F}_T$ on X and a divisor

$$D = \sum \alpha_k x_k$$

on X whose coefficients α_k are coweights, we may construct a new T-bundle $\mathcal{F}_T(D)$ which is defined by the property that for every weight μ, there is an equality

$$L^\mu_{\mathcal{F}_T(D)} = L^\mu_{\mathcal{F}_T}(\sum \langle \alpha_k, \mu \rangle x_k).$$

We will say that $\mathcal{F}_T$ has degree $\alpha \in \Lambda_G$ if for every weight μ,

$$\deg(L^\mu_{\mathcal{F}_T}) = \langle \alpha, \mu \rangle.$$

Equivalently, $\mathcal{F}_T$ has degree α if and only if $\mathcal{F}_T = \mathcal{F}_T^0(D)$, where $\mathcal{F}_T^0$ is the trivial T-bundle and D is a divisor whose coefficients sum to α. This discussion is recorded in the following proposition.

Proposition 2.23. $\pi_0(\operatorname{Bun}_B^L(X)) \simeq \pi_0(\operatorname{Bun}_B^D(X)) \simeq \pi_0(\operatorname{Bun}_B(X)) \simeq \pi_0(\operatorname{Bun}_T(X)) \simeq \Lambda_G.$

To compute the dimension of irreducible components of $\mathcal{N}$, we will need to first compute the dimension of the connected components of $\operatorname{Bun}_B(X)$. The following proposition is similar to [FM99, 3.5].

Proposition 2.24. *Let $\operatorname{Bun}_{B,\alpha}(X)$ denote the connected component of $\operatorname{Bun}_B(X)$ corresponding to $\alpha \in \Lambda_G$. Then the dimension of $\operatorname{Bun}_{B,\alpha}(X)$ is given by $-2|\alpha| + \dim(B)(g - 1)$.*

Proof. Since $\operatorname{Bun}_B(X)$ is a smooth stack, to compute the dimension of $\operatorname{Bun}_{B,\alpha}(X)$ it suffices to pick any B-bundle $\mathcal{F}_B$ on X of degree α and compute the dimension of the naive tangent complex[6] at $\mathcal{F}_B$. By standard deformation theory,

$$\dim_{\mathcal{F}_B}(\operatorname{Bun}_{B,\alpha}(X)) = -\chi(X, \mathfrak{b}_{\mathcal{F}_B}).$$

---

[6]See Section 2.4 for more details.

Since $\mathfrak{b}_{\mathcal{F}_B}$ a vector bundle of rank $\dim(B)$, it suffices by Riemann-Roch to compute the degree of $\mathfrak{b}_{\mathcal{F}_B}$. To this end, we may assume that $\mathcal{F}_B$ has a T-reduction coming from some T-bundle $\mathcal{F}_T$ of degree α. Then $\mathfrak{b}_{\mathcal{F}_B} = \mathcal{F}_T \overset{T}{\times} \mathfrak{b}$. Since

$$\mathfrak{b} = \bigoplus_{\theta \in \mathcal{R}^+} \mathfrak{g}_\theta$$

is a direct sum of positive root spaces, we see that

$$\deg(\mathcal{F}_T \overset{T}{\times} \mathfrak{b}) = \sum_{\theta \in \mathcal{R}^+} \langle \alpha, \theta \rangle = 2|\alpha|.$$

$\square$

Corollary 2.25. *The dimension of $\mathrm{Bun}_{B,\alpha}^D(X)$ and of $\mathrm{Bun}_{B,\alpha}^L(X)$ is given by $-2|\alpha|+\dim(B)(g-1)$.*

Proof. This is a direct consequence of the density statement of Proposition 2.22 and from Proposition 2.24. $\square$

2.4. **Geometry of linear systems on a curve.** In this section we will review some classic results about line bundles and linear systems on curves. The main reference for this section is [ACGH85]. We will assume that the genus of X is at least two. We will largely be concerned with the following two classical varieties (and their stack-theoretic counterparts).

$$W_d^r(X) = \{\lambda \in \mathrm{Pic}_d(X) : h^0(X,\lambda) \geq r + 1\}$$
$$G_d^r(X) = \{g_d^r\text{'s on } X\}$$

Recall that g_d^r is the classical notation for a degree d, rank r linear system on X. Therefore, in less concise notation, $G_d^r(X)$ is the variety of pairs (λ, V) where $\lambda \in \mathrm{Pic}_d(X)$ and $V \subset H^0(X,\lambda)$ is a subspace of dimension $r + 1$. For the purposes of global Springer theory, we will be especially interested in $G_d^0(X)$. The two varieties $W_d^r(X)$ and $G_d^r(X)$ are related by the fact that the scheme-theoretic image of $G_d^r(X)$ under the projection map $G_d^r(X) \to \mathrm{Pic}_d(X)$ is precisely $W_d^r(X)$. The next result summarizes the crucial geometric properties of $W_d^0(X)$ and $G_d^0(X)$.

Theorem 2.26. *(i) $G_d^0(X)$ is smooth of dimension d for all d.*
(ii) $W_d^0(X)$ is reduced, irreducible, normal, and Cohen-Macaulay of dimension d. If $d < g$, then the singular locus of $W_d^0(X)$ is $W_d^1(X)$.

Proof. See [ACGH85, Cor. 4.5]. $\square$

The assumption that $d < g$ in Theorem 2.26 is present simply because $W_d^0(X)$ is smooth whenever $d \geq g$. Indeed, given $\lambda \in \mathrm{Pic}_d(X)$ with $d \geq g$, the Riemann-Roch formula implies that

$$\chi(X,\lambda) = d + 1 - g \geq 1.$$

Therefore $W_d^0(X) = \mathrm{Pic}_d(X)$ is smooth.

In Section 3.5, we will need variants of $W_d^0(X)$ and $G_d^0(X)$, which we now discuss. First, recall that the Picard stack is obtained by taking the stack quotient $\mathrm{Pic}(X)/\mathbb{G}_m$ of the Picard variety by a trivial action of $\mathbb{G}_m$. To avoid confusion, the Picard stack will be denoted $\mathrm{Bun}_{\mathbb{G}_m}(X)$. There are then stack-theoretic versions of $W_d^r(X)$ and $G_d^r(X)$:

$$\mathcal{W}_d^r(X) = W_d^r(X)/\mathbb{G}_m$$
$$\mathcal{G}_d^r(X) = G_d^r(X)/\mathbb{G}_m$$

To have a good understanding of the partial global Springer resolution defined in Section 3.1, it is necessary to give a precise formulation of the S-points of $\mathcal{W}_d^r(X)$ and $\mathcal{G}_d^r(X)$, as described in [ACGH85, Sec. 4.3]. In order to do so, we now review basic properties of Fitting ideals, which may

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be found in [MW84, Appendix]. To begin, assume that M is a finitely presented module over a commutative ring R. Given a presentation

$$R^{\oplus a} \xrightarrow{A} R^{\oplus b} \to M \to 0,$$

define, for $h \geq 0$, the hth *Fitting ideal* of M to be the ideal of R generated by the $(b-h) \times (b-h)$ minors of A. It is denoted $F^h(M)$ or $F_R^h(M)$. We adopt the convention that $F^h(M) = R$ if $a < b - h$. The following properties of $F^h(M)$ are well-known.

(1) $F^h(M)$ is independent of the presentation chosen for M.
(2) If N is also an R-module, then $F^h(M \oplus N) = F^h(M)F^h(N)$.
(3) Fitting ideals are stable under base change in the sense that if A is an R-algebra, then $F_A(M \underset{R}{\otimes} A) = F_R(M)A$.

More generally, suppose that $\mathcal{F}$ is a coherent $\mathcal{O}_S$-module for some scheme S. Then the hth Fitting ideal $F^h(\mathcal{F})$ is defined to be the ideal sheaf of S defined locally on affine subvarieties of S as the Fitting ideal of the corresponding finitely presented module. In the language of coherent sheaves, properties (1) through (3) above are translated as follows.

(1) $F^h(\mathcal{F})$ is independent of the local presentation for $\mathcal{F}$.
(2) If $\mathcal{G}$ is also a coherent $\mathcal{O}_S$-module, then $F(\mathcal{F} \oplus \mathcal{G}) = F(\mathcal{F})F(\mathcal{G})$.
(3) If T is an S-scheme given by $f : T \to S$, then $F_T(f^*\mathcal{F}) = f^{-1}F_S(\mathcal{F}) \cdot \mathcal{O}_T$.

Given $\mathcal{F}$ as above, the *Fitting rank* of $\mathcal{F}$ is defined to be the largest integer h such that $F^h(\mathcal{F}) = 0$. An S-point of $\mathcal{W}_d^r(X)$ is then defined to be a degree d line bundle $\lambda \in \text{Bun}_{\mathbb{G}_m, d}(X)(S)$ such that the Fitting rank of $R^1 p_* \lambda$ is at least $g - d + r$. While this may appear quite different from the initial definition of the $\mathbb{C}$-points of $W_d^r(X)$, note that $\lambda \in \mathcal{W}_d^r(X)(S)$ implies that $h^1(X_s, \lambda_s) \geq g - d + r$ for each $s \in S$. By Riemann-Roch, this equivalent to $h^0(X_s, \lambda_s) \geq r + 1$ for each $s \in S$.

An S-point of $\mathcal{G}_d^r(X)$ is defined to be a pair (λ, V) where $\lambda \in \text{Bun}_{\mathbb{G}_m, d}(X)(S)$ and V is a locally free subsheaf of $p_* \lambda$ of rank r which remains a subsheaf under arbitrary base change.

Rather than considering $\mathcal{G}_d^0(X)$, we will instead consider the moduli stack of pairs (λ, s) where λ is a line bundle on X of degree d, $s \in H^0(X, \lambda)$, and $h^0(X, \lambda) \geq 1$.

Since the moduli of pairs (λ, s) with $s \in H^0(X, \lambda)$ is a relative affine cone of $\mathcal{G}_d^0(X)$ over the fibers of $\text{Bun}_{\mathbb{G}_m}(X)$, we will let $C\mathcal{G}(X)$ denote the moduli stack whose S-points consist of all pairs (λ, s) in which $\lambda \in \mathcal{W}_d^0(X)(S)$ and $s \in H^0(X_S, \lambda)$. $C\mathcal{G}_d(X)$ will denote the connected component of degree d.

We now investigate the extent to which the results of Theorem 2.26 apply to the geometry of $\mathcal{W}_d^0(X)$ and $C\mathcal{G}_d(X)$. The first obvious observation is that $W_d^r(X)$ and $G_d^r(X)$ are atlases for $\mathcal{W}_d^r(X)$ and $\mathcal{G}_d^r(X)$, respectively. Therefore we only need to consider $C\mathcal{G}_d(X)$.

To begin, let us review the tangent space computations of $G_d^r(X)$ and $W_d^r(X)$ found in [ACGH85, Sec. 4.4]. Let

$$v : G_d^r(X) \to \text{Pic}_d(X)$$

denote the projection map. Then, given $(\lambda, V) \in G_d^r(X)$, we have the following exact sequence of tangent spaces

(2.3) $$0 \to T_{(\lambda, V)}(v^{-1}(\lambda)) \to T_{(\lambda, V)}(G_d^r(X)) \xrightarrow{v_*} T_\lambda(\text{Pic}_d(X)).$$

In (2.3), we are interested in computing $T_{(\lambda, V)}(G_d^r(X))$, which means that we need to compute both $T_{(\lambda, V)}(v^{-1}(\lambda))$ and the image of v_*.

$T_{(\lambda, V)}(v^{-1}(\lambda))$ corresponds to first-order deformations of the pair (λ, V) in which λ is deformed trivially. Therefore, the first-order deformations coincide with the first-order deformations of V as an element of the Grassmannian $Gr_{r+1}(H^0(X, \lambda))$. We conclude that

$$T_{(\lambda, V)}(v^{-1}(\lambda)) \simeq \text{Hom}(V, H^0(X, \lambda)/V).$$

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The image of v_* corresponds to those first-order deformations of λ such that the subspace $V \subset H^0(X, \lambda)$ may be deformed in a compatible way. Recall first the well-known identification

$$T_\lambda(\mathrm{Pic}_d(X)) \simeq H^1(X, \mathcal{O}_X).$$

To compute the image of v_*, it will be most convenient to fix a section $s \in H^0(X, \lambda)$ and a class $\phi \in H^1(X, \mathcal{O}_X)$, and to find necessary conditions so that there exists a section s_ϕ of the deformation λ_ϕ of λ such that the restriction of s_ϕ to X is s. We may view λ_ϕ as sitting in a short exact sequence

(2.4) $$0 \to \lambda \to \lambda_\phi \to \lambda \to 0$$

of $\mathcal{O}_X$-modules, in which the additional structure of $\mathcal{O}_{X[\epsilon]}$-module comes from composing the projection $\lambda_\phi \to \lambda$ with the inclusion $\lambda \to \lambda_\phi$. Then (2.4) yields a long exact sequence

$$\ldots \to H^0(X, \lambda_\phi) \xrightarrow{r_\phi} H^0(X, \lambda) \xrightarrow{\delta_\phi} H^1(X, \lambda) \to \ldots,$$

in which compatible deformations of s correspond to the inverse image $r_\phi^{-1}(s)$. Therefore, $\phi \in H^1(X, \mathcal{O}_X)$ is in the image of v_* if and only if s is in the image of r_ϕ. Since $\mathrm{im}(r_\phi) = \ker(\delta_\phi)$ and δ_ϕ is given by

$$\delta_\phi(t) = \phi \cup t,$$

the image of v_* is

(2.5) $$\{\phi \in H^1(X, \mathcal{O}_X) : \phi \cup s = 0 \in H^1(X, \lambda) \text{ for all } s \in V\}.$$

We can use the tangent space computation for $G_d^r(X)$ to also compute the tangent spaces for $W_d^r(X)$. Since the restriction of the projection

$$G_d^r(X) \to W_d^r(X)$$

is an isomorphism away from $W_d^{r+1}(X)$, we see that

(2.6) $$T_\lambda(W_d^r(X)) = \{\phi \in H^1(X, \mathcal{O}_X) : \phi \cup s = 0 \text{ for all } s \in H^0(X, \lambda)\}, \text{ if } \lambda \notin W_d^{r+1}(X).$$

Furthermore, general results on determinantal varieties imply that (see [ACGH85, Sec. 2.2])

(2.7) $$T_\lambda(W_d^r(X)) = H^1(X, \mathcal{O}_X), \text{ if } \lambda \in W_d^{r+1}(X).$$

The above argument will need to be slightly altered to work for $C\mathcal{G}_d(X)$. The first issue is that $G_d^r(X)$ is a scheme, whereas $C\mathcal{G}_d(X)$ is a stack. We will therefore have to replace tangent spaces with naive tangent complexes[7], and to replace the dimension of the tangent space by the Euler characteristic of the naive tangent complex. Then, just as an equidimensional scheme of dimension d is smooth at a point x if and only if the dimension of the tangent space at x is d, an equidimensional stack of dimension d is smooth at a point x if and only if the Euler characteristic of the naive tangent complex at x is d.

However, these differences are actually quite mild. If F is a sheaf of groupoids and x is a $\mathbb{C}$-point of F, then the naive tangent complex $\mathcal{T}_x(F)$ is a 2-term complex lying in degrees -1 and 0 such that

$$H^0(\mathcal{T}_x(F)) = \{\text{isomorphism classes of first-order deformations of } x\},$$

$$H^{-1}(\mathcal{T}_x(F)) = \mathrm{Lie}(\mathrm{Aut}(x)).$$

We are now in a good position to verify the following geometric properties of $C\mathcal{G}_d(X)$.

Proposition 2.27. $C\mathcal{G}_d(X)$ *is irreducible of dimension d.*

---

[7]We use the term "naive tangent complex" because it can be defined for any sheaf of groupoids, and should not be confused with the tangent complex of L. Illusie ([Ill71]).

(1) *If $d < g$, a $\mathbb{C}$-point (λ, s) of $CG_d(X)$ is smooth if and only if either $s \neq 0$, or $s = 0$ and $h^0(X, \lambda) = 1$.*

(2) *If $g \leq d \leq 2g - 2$, a $\mathbb{C}$-point (λ, s) of $CG_d(X)$ is smooth if and only if either $s \neq 0$, or $s = 0$ and $H^1(X, \lambda) = 0$.*

(3) *Finally, if $d > 2g - 2$, then $CG_d(X)$ is smooth.*

Proof. Let U denote the complement of the zero section inside $CG_d(X)$. Then the induced map $U \to G_d^0(X)$ is smooth of relative dimension 1. Since $\dim(G_d^0(X)) = d$, the dimension of $G_d^0(X)$ is $d - 1$. Therefore

$$\dim(U) = d.$$

Since U is an irreducible (in fact, smooth) and dense open substack of $CG_d(X)$, it follows that $CG_d(X)$ is irreducible of dimension d.

To determine the smooth points of $CG_d(X)$, fix a $\mathbb{C}$-point (λ, s) of $CG_d(X)$. The computation of $H^0(\mathcal{T}_{(\lambda,s)}(CG_d(X)))$ will be quite similar to the tangent space computations for $G_d^r(X)$. In fact, we can deduce from analogs of (2.3) and (2.5) that a first-order deformation of (λ, s) is given by the following data.

(1) A section $t \in H^0(X, \lambda)$.

(2) A class $\phi \in H^1(X, \mathcal{O}_X)$ such that $\phi \cup s = 0$. Furthermore, if $s = 0$, we must have $\phi \in T_\lambda(W_d^0(X))$.

In order to find the isomorphism classes of first-order deformations, note that an automorphism of λ_ϕ which preserves λ is given by some $1 + a\epsilon \in \mathbb{C}[\epsilon]$. Then an isomorphism between $(\lambda_\phi, s + t_1\epsilon)$ and $(\lambda_\phi, s + t_2\epsilon)$ is given by an element $1 + a\epsilon$ such that

$$(1 + a\epsilon)(s + t_1\epsilon) = s + t_2\epsilon.$$

Therefore, $(\lambda_\phi, s + t_1\epsilon) \simeq (\lambda_\phi, s + t_2\epsilon)$ if and only if

$$t_1 + as = t_2$$

for some $a \in \mathbb{C}$. If $s \neq 0$, then a given first-order deformation of (λ, s) therefore lives in a 1-dimensional family of isomorphic deformations. On the other hand, if $s = 0$, then any first-order deformation of (λ, s) has no non-trivial automorphisms.

To completely determine $H^0(\mathcal{T}_{(\lambda,s)}(CG_d(X)))$, we now classify those $\phi \in H^1(X, \mathcal{O}_X)$ such that $\phi \cup s = 0$ and such that $\phi \in T_\lambda(W_d^0(X))$. If $s \neq 0$, then the computation is straightforward. In this case, s determines a short exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{s} \lambda \to T \to 0$$

in which the quotient T is torsion. Therefore $H^1(X, T) = 0$, and the induced long exact sequence ends with

$$H^1(X, \mathcal{O}_X) \to H^1(X, \lambda) \to 0.$$
$$\alpha \mapsto \alpha \cup s$$

Surjectivity shows that the dimension of $\{\phi \in H^1(X, \mathcal{O}_X) : \phi \cup s = 0\}$ is $g - h^1(X, \lambda)$.

If $s = 0$, then (2.6) and (2.7) classify those $\phi \in H^1(X, \mathcal{O}_X)$ corresponding to first-order deformations of $(\lambda, 0)$. Let us now compute $h^0(\mathcal{T}_{(\lambda,s)}(CG_d(X)))$.

(1) Suppose that $s \neq 0$. Then

$$h^0(\mathcal{T}_{(\lambda,s)}(CG_d(X))) = h^0(X, \lambda) - 1 + g - h^1(X, \lambda) = d.$$

(2) Suppose that $d < g$ and that $s = 0$. Then

$$h^0(\mathcal{T}_{(\lambda,0)}(CG_d(X))) = h^0(X, \lambda) + g - h^1(X, \lambda) = d + 1, \text{ if } h^0(X, \lambda) = 1,$$
$$h^0(\mathcal{T}_{(\lambda,0)}(CG_d(X))) = h^0(X, \lambda) + g, \text{ if } h^0(X, \lambda) > 1.$$

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(3) Suppose that $g \leq d \leq 2g - 2$ and that $s = 0$. Then
$$h^0(\mathcal{T}_{(\lambda,s)}(C\mathcal{G}_d(X))) = h^0(X,\lambda) + g.$$

We note that $h^0(X,\lambda) + g = d + 1$ if and only if $h^1(X,\lambda) = 0$.

The calculation of $h^1(\mathcal{T}_{(\lambda,s)}(C\mathcal{G}_d(X)))$ is much simpler. An automorphism of (λ, s) is given by a scalar $a \in \mathbb{C}^\times$ such that $as = s$. Therefore if $s \neq 0$, then (λ, s) has no non-trivial automorphisms, while if $s = 0$, then $(\lambda, 0)$ has automorphism group $\mathbb{C}^\times$.

It is now clear that, for $d \leq 2g - 2$,
$$\chi(\mathcal{T}_{(\lambda,s)}(C\mathcal{G}_d(X)) = d$$

if and only if

(1) $s \neq 0$, or
(2) $s = 0$, $d < g$, and $h^0(X,\lambda) = 1$, or
(3) $s = 0$, $g \leq d \leq 2g - 2$, and $h^1(X,\lambda) = 0$.

Lastly, to see that $C\mathcal{G}_d(X)$ is smooth whenever $d > 2g - 2$, it is easiest to note that $H^1(X,\lambda) = 0$ in this case and that $C\mathcal{G}_d(X)$ is therefore a vector bundle of rank $d + 1 - g$ over $\mathrm{Bun}_{\mathbb{G}_m}(X)$. $\qquad\square$

3. A GLOBAL SPRINGER RESOLUTION

In this main section of the paper, we specialize to $G = SL_2$. Correspondingly, the associated Laumon/Drinfeld compactification will be denoted by $\overline{\mathrm{Bun}}_B(X)$. Furthermore, we will assume that $\deg(\mathcal{L})$ is even and that $\deg(\mathcal{L}) \geq 2g$ (with the exception of Section 3.6). The same assumptions on the degree of $\mathcal{L}$ may also be found in [Ngô10] and [Yun11].

We begin by constructing a partial global analog of the Springer resolution, denoted $\widehat{\mathcal{N}}$. It turns out that the geometry of $\widehat{\mathcal{N}}$ is closely tied to the geometry of line bundles on curves. For this reason, as the genus of the curve increases, $\widehat{\mathcal{N}}$ becomes more complicated to understand. In fact, $\widehat{\mathcal{N}}$ is only smooth (and hence an actual resolution) when the genus of the curve is 0 or 1. For this reason, after showing that $\widehat{\mathcal{N}}$ is proper and birational over $\mathcal{N}$ in Section 3.2, we will begin by describing $\widehat{\mathcal{N}}$ when $X = \mathbb{P}^1$, where everything is as simple as possible. This situation is illuminating because many of the main ideas are presented without complications arising from the particular geometry of the curve. After this, we will describe $\widehat{\mathcal{N}}$ when X is an elliptic curve. This provides a useful bridge between the genus 0 case and the higher genus cases. While we still obtain an honest resolution when X is an elliptic curve, some extra care must be given in regards to the particular geometry of the curve. Finally, we will then study $\widehat{\mathcal{N}}$ in the case of curves of genus greater than 1. While $\widehat{\mathcal{N}}$ is not smooth in this case, its geometry is intimately related to that of $C\mathcal{G}(X)$. We will then be able to resolve $\widehat{\mathcal{N}}$ further, and obtain a stack $\widetilde{\mathcal{N}}$ which is a resolution of singularities of $\mathcal{N}$. Finally, we end by discussing twisting bundles $\mathcal{L}$ of lower degree, as well as the stable locus of $\mathcal{N}$.

3.1. Construction of a global Springer resolution.
The purpose of this section is to provide definitions and basic results which will be applicable regardless of the genus of X.

Definition 3.1. Suppose that $(\lambda \subset E) \in \overline{\mathrm{Bun}}_B(X)(\mathbb{C})$. Then $E/\lambda \simeq F \oplus T$ where F is a line bundle and T is torsion. The unique line bundle $\widetilde{\lambda} \supset \lambda$ such that $E/\widetilde{\lambda} \simeq F$ is called the *normalization* of $\lambda \subset E$. The effective divisor on which the torsion sheaf $T \simeq \widetilde{\lambda}/\lambda$ is supported is called the *defect* of $\lambda \subset E$. The defect will be denoted by $\mathrm{def}(\lambda \subset E)$, or by $\mathrm{def}(\lambda)$.

In order to define a candidate for the global Springer resolution, it is necessary to extend the definition of defect to arbitrary S-points of $\overline{\mathrm{Bun}}_B(X)$. That is, given an S-point $\lambda \subset E$ of $\overline{\mathrm{Bun}}_B(X)$, we seek a relative effective Cartier divisor $\mathrm{def}(\lambda)$ on X_S which measures the failure of λ to be a

subbundle of E, and such that $\mathrm{def}(\lambda)_s = \mathrm{def}(\lambda_s)$ for all geometric points $s \in S$. Just as when $S = \mathrm{Spec}(\mathbb{C})$, the divisor $\mathrm{def}(\lambda)$ will be referred to as the *defect* of $\lambda \subset E$.

The solution to this problem is provided by Fitting ideals, which were reviewed in Section 2.4. Before stating the result, there is one further basic property of Fitting ideals not previously mentioned. As before, both module and sheaf theoretic versions are given.

(1) If R is a discrete valuation ring with maximal ideal $\mathfrak{m}$ and M is a finitely generated R-module, then $F^0(M) = \mathfrak{m}^{\ell(M)}$, where $\ell(M)$ denotes the length of M as an R-module.

(2) If X is a smooth curve and there is an exact sequence

$$0 \to \lambda \to E \to Q \to 0$$

with λ a line bundle and E a vector bundle of rank r, then $F^{r-1}(Q)$ is the ideal sheaf of the divisor on which the torsion part of Q is supported. In other words, $F^{r-1}(Q)$ is the ideal sheaf of the defect of $\lambda \subset E$.

The following proposition shows that if $(\lambda \subset E) \in \overline{\mathrm{Bun}}_B(X)(S)$, then the first Fitting ideal of the quotient is the sought after generalization of the defect to S-points of $\overline{\mathrm{Bun}}_B(X)$.

Proposition 3.2. *Given a scheme S, suppose that*

$$0 \to \lambda \to E \to Q \to 0$$

is an exact sequence of coherent sheaves on X_S where λ is a line bundle and E is a vector bundle of rank r, and such that Q is S-flat. Then there exists a relative effect Cartier divisor $\mathrm{def}(\lambda)$ on X_S over S which measures the failure of λ to be a subbundle of E, and such that the fiber $\mathrm{def}(\lambda)_s$ coincides with the defect of the fiber $\mathrm{def}(\lambda_s)$ for every geometric point $s \in S$.

Proof. We claim that we may define $\mathrm{def}(\lambda)$ to be the closed subscheme of X_S defined by the ideal sheaf $F^{r-1}(Q)$. To show that $\mathrm{def}(\lambda)$ is a relative Cartier divisor on X_S, it suffices to prove, using [KM85, Lem. 1.2.3], that $\mathrm{def}(\lambda)$ is finite and flat over S. The closed subscheme $\mathrm{def}(\lambda)$ is proper over S because X_S is proper over S. Furthermore, $\mathrm{def}(\lambda)$ is quasi-finite over S because $\mathrm{def}(\lambda)_s = \mathrm{def}(\lambda_s)$ for every $s \in S$. It follows that $\mathrm{def}(\lambda)$ is finite over S.

To show that $\mathrm{def}(\lambda)$ is S-flat, we use the local criterion for flatness ([Eis95, Thm. 6.8]) and the locally free resolution of Q given by

$$0 \to \lambda \to E \to Q \to 0.$$

Since flatness is an open property, it suffices to check the local criterion of flatness only at closed points of S. Let $s \in S$ be a closed point, and pick $x \in X_S$ such that $p(x) = s$. Then we need to check that

$$(3.1) \qquad\qquad \mathrm{Tor}_1^{i_s^* \mathcal{O}_S}(\mathcal{O}_{S,s}, i_x^* \mathcal{O}_{\mathrm{def}(\lambda)}) = 0.$$

Then consider the corresponding map on the flat resolution of Q,

$$f_s : i_x^* \lambda \otimes \mathcal{O}_{S,s} \to i_x^* E \otimes \mathcal{O}_{S,s}.$$

The map f_s is injective due to the assumption that Q is S-flat. Furthermore, the ideal sheaf of $\mathrm{def}(\lambda)$ at s is defined by the ideal generated by the entries of f_s. Since X is a smooth curve, the entries of f_s are elements of a discrete valuation ring. Therefore, since f_s is injective, the ideal generated by the entries of f_s is a principal ideal generated by some t^k where t is a uniformizing parameter and $k \geq 0$. Therefore the desired equality in (3.1) holds.

We have thus shown that $\mathrm{def}(\lambda)$ is a relative effective Cartier divisor on X_S over S which coincides with $\mathrm{def}(\lambda_s)$ over the geometric points of S. Finally, the fiber-by-fiber criterion for flatness over X_S ([AK70, Cor V.3.6]) implies that $\mathrm{def}(\lambda)$ is the locus where Q is not locally free (equivalently, where λ fails to be a subbundle of E). $\qquad\square$

We are now ready to define the partial global Springer resolution.

Definition 3.3. The *partial global Springer resolution* $\widehat{\mathcal{N}}_{X,G,\mathcal{L}}$ (or simply $\widehat{\mathcal{N}}$) is the moduli stack whose S-points consist of

$$((E,\varphi),\lambda \subset E) \in \mathcal{N}(S) \underset{\mathrm{Bun}_G(X)(S)}{\times} \overline{\mathrm{Bun}}_{B,d}(X)(S)$$

such that the following conditions hold.

(1) $\lambda \subset \ker(\varphi)$.
(2) $\mathrm{im}(\varphi) \subset (\lambda \otimes \mathcal{L})(-\mathrm{def}(\lambda))$.
(3) $\lambda^{\otimes 2} \otimes \pi^*\mathcal{L} \in \mathcal{W}^0(X)(S)$.

The definition of $\widehat{\mathcal{N}}$ deserves some explanation. Condition (1) is motivated by the ordinary Springer resolution, which consists of flags preserved by a nilpotent endomorphism. It also causes $\widehat{\mathcal{N}}$ to be a closed substack of $T^*\overline{\mathrm{Bun}}_B(X)$ when $\mathcal{L} = \omega_X$. Condition (2) can be motivated in a couple of ways. Since φ is nilpotent, $\mathrm{im}(\varphi) \subset \ker(\varphi) \otimes \mathcal{L}$, and condition (2) can be viewed as a strengthening of this. Indeed, in the ideal situation where φ is generically regular and $\lambda = \ker(\varphi)$, the defect is zero and nothing new occurs. In this case, we are then largely able to understand (E,φ) through the data of λ and the induced map $\overline{\varphi} : \mathrm{im}(\varphi) \to \lambda \otimes \mathcal{L}$. Since λ is a subbundle, $\mathrm{im}(\varphi) = \lambda^{-1}$, and $\overline{\varphi}$ can be viewed as a global section of $\lambda^{\otimes 2} \otimes \mathcal{L}$.

In a setting where $\lambda \subset E$ is not a subbundle, we would still like to have the data of $\overline{\varphi} \in H^0(X, \lambda^{\otimes 2} \otimes \mathcal{L})$, and this is precisely what condition (2) provides. This is desirable for a couple of reasons. The first is that without this condition, $\widehat{\mathcal{N}}$ would not be birationally equivalent to $\mathcal{N}$, as most of the fibers would be much too large. Secondly, condition (2) makes it possible for $\widehat{\mathcal{N}}$ to be a vector bundle over $\overline{\mathrm{Bun}}_B(X)$, much like $\widetilde{\mathcal{N}ilp}$ is a vector bundle over G/B. For certain degrees of λ, this will not be true due to the non-constancy of $h^0(X, \lambda^{\otimes 2} \otimes \mathcal{L})$, but even in this situation condition (2) still allows us to understand $\widehat{\mathcal{N}}$ in terms of $C\mathcal{G}(X)$.

There is also a motivation for condition (2) coming from the notion of the irregularity of the nilpotent Higgs field φ. Intuitively speaking, the irregularity of φ, denoted $\mathrm{irr}(\varphi)$, is the divisor which measures the failure of a generically regular φ to be regular. Since $G = SL_2$, regularity of φ is equivalent to φ having rank 1 everywhere, which means that $\mathrm{irr}(\varphi)$ is roughly the divisor of zeroes of φ. More precisely, if φ is generically regular, $\mathrm{irr}(\varphi) = \mathrm{def}(\mathrm{im}(\varphi) \subset E \otimes \mathcal{L})$. Then condition (2) can be viewed as saying that $2\,\mathrm{def}(\lambda) \subset \mathrm{irr}(\varphi)$ for generically regular φ. This simply says that there should be a relationship between the failure of λ to be a subbundle of E and the failure of φ to be regular.

Example 3.4. Let $X = \mathbb{P}^1$ and $\mathcal{L} = \mathcal{O}(2)$. Consider the nilpotent endomorphism $\varphi : \mathcal{O} \oplus \mathcal{O} \to (\mathcal{O} \oplus \mathcal{O}) \otimes \mathcal{O}(2)$ given by $\varphi = \begin{pmatrix} 0 & z^2 \\ 0 & 0 \end{pmatrix}$. It turns out that $(\mathcal{O} \oplus \mathcal{O}, \varphi)$ lies in the intersection of two irreducible components of $\mathcal{N}(\mathbb{C})$. Without condition (2) of Definition 3.3, the fiber of $(\mathcal{O} \oplus \mathcal{O}, \varphi)$ in $\widehat{\mathcal{N}}_1$ would correspond to all inclusions $\mathcal{O}(-1) \subset \mathcal{O} \oplus \mathcal{O}$ of the form $\begin{pmatrix} s \\ 0 \end{pmatrix}$ with $s \in H^0(\mathcal{O}(1))$, and hence would be isomorphic to $\mathbb{P}^1$. Furthermore, this example generalizes to any $(\mathcal{O} \oplus \mathcal{O}, \varphi)$ such that $\mathrm{irr}(\varphi)$ is a skyscraper sheaf supported on $2x$ for some $x \in X$. However, condition (2) of Definition 3.3 implies that the section $s \in H^0(\mathcal{O}(1))$ must be of the form $\begin{pmatrix} z \\ 0 \end{pmatrix}$, and hence the fiber consists of a single point in this connected component.

Finally, condition (3) of Definition 3.3 is there for a couple of reasons. First, it dictates that $\deg(\lambda) \geq -\frac{1}{2}\deg(\mathcal{L})$. Therefore, the connected components of $\widehat{\mathcal{N}}$ are indexed by:

(1) All integers $d > -\frac{1}{2}\deg(\mathcal{L})$.
(2) All 2^{2g} square roots of $\mathcal{L}^{-1}$.

For each such integer d, the corresponding connected component will be denoted $\widehat{N}_d$. For convenience, this notation will also be used when $d = -\frac{1}{2}\deg(\mathcal{L})$ to denote the union of connected components for which $\deg(\lambda) = -\frac{1}{2}\deg(\mathcal{L})$. In other words,

$$\widehat{\mathcal{N}}_{-\frac{1}{2}\deg(\mathcal{L})} = \bigcup_{L}\widehat{N}_L,$$

where $L^{\otimes 2} = \mathcal{L}^{-1}$.

Lastly, requiring that $\lambda^{\otimes 2} \otimes \mathcal{L} \in \mathcal{W}^0(X)$ prevents superfluous Higgs bundles from appearing in the fibers of the projection map $\widehat{\mathcal{N}} \to \overline{\mathrm{Bun}}_B(X)$ in the sense that any fiber will necessarily contain nilpotent Higgs bundles with a nonzero Higgs field.

3.2. **Properness and birationality of $\widehat{\mathcal{N}} \to \mathcal{N}$.** The purpose of this section is to show that the projection

$$\mu : \widehat{\mathcal{N}} \to \mathcal{N}$$

is both proper and a birational equivalence. Issues of smoothness and irreducibility of $\widehat{\mathcal{N}}$ will be addressed in subsequent sections.

Proposition 3.5. μ *is proper.*

Proof. Consider the following diagram.

$$
\begin{array}{ccc}
\overline{\mathrm{Bun}}_B(X) & \longleftarrow & \overline{\mathrm{Bun}}_B(X) \underset{\mathrm{Bun}_G(X)}{\times} \mathcal{N} \\
\downarrow & & \downarrow \\
\mathrm{Bun}_G(X) & \longleftarrow & \mathcal{N}
\end{array}
$$

Since $\overline{\mathrm{Bun}}_B(X) \to \mathrm{Bun}_G(X)$ is proper and $\mathcal{N}$ is a closed substack of $\mathcal{M}$, the pull-back map

$$\overline{\mathrm{Bun}}_B(X) \underset{\mathrm{Bun}_G(X)}{\times} \mathcal{N} \to \mathcal{N}$$

is also proper. Therefore in order to show that $\widehat{\mathcal{N}} \to \mathcal{N}$ is proper, it suffices to note that $\widehat{\mathcal{N}}$ is a closed substack of $\overline{\mathrm{Bun}}_B(X) \underset{\mathrm{Bun}_G(X)}{\times} \mathcal{N}$. $\qquad\square$

To show that μ is a birational equivalence between $\widehat{\mathcal{N}}$ and $\mathcal{N}$, we will define an open substack of $\mathcal{N}$ over which μ is an isomorphism, which will be referred to as the locus of *globally regular* nilpotent Higgs bundles. To define this substack, first recall the algebraic stack $Coh_{X,0}$ as defined in [Lau87]. $Coh_{X,0}$ is the moduli stack of finite length coherent sheaves on X. Its connected components are given by $Coh_{X,0}^m$, which denotes the moduli stack of coherent sheaves on X of length m.

For a fixed m, each $Coh_{X,0}^m$ is stratified by locally closed substacks $Coh_{X,0}^{(m_1,\ldots,m_k)}$, where $(m_1,\ldots,m_k)$ is a partition of m. The unique open stratum corresponds to the the partition (m). In this case, $Coh_{X,0}^{(m)}$ parameterizes those length m coherent sheaves on X which are supported on a divisor of the form $\sum_{i=1}^{m} x_i$, in which the x_i are distinct points of X.

Let $\mathcal{N}^{\mathrm{gen,reg}}$ denote the open locus of generically regular elements of $\mathcal{N}$. Then there is a map

$$\alpha : \mathcal{N}^{\mathrm{gen,reg}} \to Coh_{X,0}$$

which is defined as follows. Recalling that for a generically regular (E,φ), the irregularity $\mathrm{irr}(\varphi)$ is the divisor which is defined to be the defect of $\mathrm{im}(\varphi) \subset E \otimes \mathcal{L}$, the map α is defined by

$$\alpha((E,\varphi)) = \mathcal{O}_{\mathrm{irr}(\varphi)}.$$

Finally, the globally regular substack $\mathcal{N}^{\mathrm{gl,reg}}$ of $\mathcal{N}$ is defined to be

$$\alpha^{-1}\left(\bigcup_{m\geq 0} Coh_{X,0}^{(m)}\right).$$

$\mathcal{N}^{\mathrm{gl,reg}}$ is therefore an open substack of $\mathcal{N}^{\mathrm{gen,reg}}$ because each $Coh_{X,0}^{(m)}$ is open in $Coh_{X,0}^m$. Since $\mathcal{N}^{\mathrm{gen,reg}}$ is an open substack of $\mathcal{N}$, it follows that $\mathcal{N}^{\mathrm{gl,reg}}$ is as well.

Proposition 3.6. $\mu : \widehat{\mathcal{N}} \to \mathcal{N}$ *is a birational equivalence. More specifically, the restriction of μ to* $\mathcal{N}^{\mathrm{gl,reg}}$ *is an isomorphism. The inverse map is given by*

$$\nu : \mathcal{N}^{\mathrm{gl,reg}} \to \widehat{\mathcal{N}}$$
$$(E,\varphi) \mapsto (\ker(\varphi) \subset E, \varphi).$$

Proof. It is clear that $\mu \circ \nu = \mathrm{id}_{\mathcal{N}^{\mathrm{gl,reg}}}$. To show that $\nu \circ \mu = \mathrm{id}_{\widehat{\mathcal{N}}}$, it suffices to show that the fiber over a point $(E,\varphi) \in \mathcal{N}^{\mathrm{gl,reg}}$ consists of a single point. Suppose then that $\lambda \subset E$ is in the fiber over (E,φ). Since $2\,\mathrm{def}(\lambda) \subset \mathrm{irr}(E,\varphi)$, we must have $\mathrm{def}(\lambda) = 0$, which implies that $\lambda = \ker(\varphi)$. $\square$

3.3. A global Springer resolution in genus 0.
In this section, $X = \mathbb{P}^1$ and $\mathcal{L}$ will be any line bundle on X of even, nonnegative degree.

Let us briefly explain what aspect of $\mathbb{P}^1$ makes its global nilpotent cone so much easier to study than that of higher genus curves. Recall that on an arbitrary curve, the Euler characteristic of any line bundle is a linear function of the degree of the line bundle. On the other hand, for any line bundle on $\mathbb{P}^1$ of nonnegative degree, h^0 is a linear function of the degree of the line bundle (i.e., H^1 vanishes). This is important because for any $(\lambda \subset E) \in \overline{\mathrm{Bun}}_B(X)$ in the image of the projection from $\widehat{\mathcal{N}}$, the geometry of $\widehat{\mathcal{N}}$ will largely be controlled by $H^0(X, \lambda^{\otimes 2} \otimes \mathcal{L})$. By condition (3) of Definition 3.3, the degree of $\lambda^{\otimes 2} \otimes \mathcal{L}$ is nonnegative, and so the dimension of $H^0(X, \lambda^{\otimes 2} \otimes \mathcal{L})$ depends only on the degree of λ when $X = \mathbb{P}^1$.

Theorem 3.7. *When $X = \mathbb{P}^1$, the projection map $\mu : \widehat{\mathcal{N}} \to \mathcal{N}$ is a resolution of singularities.*

Proof. Given Proposition 3.5 and Proposition 3.6, all that is left to prove is that $\widehat{\mathcal{N}}$ is smooth. This follows from Proposition 3.8 below together with the smoothness of $\overline{\mathrm{Bun}}_B(X)$. $\square$

Proposition 3.8. *The projection map $\widehat{\mathcal{N}}_d \to \overline{\mathrm{Bun}}_{B,d}(X)$ is a vector bundle of rank $2d + \deg(\mathcal{L}) + 1$.*

Proof. For any S-point of $\overline{\mathrm{Bun}}_{B,d}(X)$, consider the following pull-back diagram.

$$
\begin{array}{ccc}
\widehat{\mathcal{N}}_{S,d} := S \underset{\overline{\mathrm{Bun}}_{B,d}(X)}{\times} \widehat{\mathcal{N}}_d & \longrightarrow & \widehat{\mathcal{N}}_d \\
\downarrow & & \downarrow \\
S & \longrightarrow & \overline{\mathrm{Bun}}_{B,d}(X)
\end{array}
$$

If $S \to \overline{\mathrm{Bun}}_{B,d}(X)$ corresponds to $\lambda \subset E$, then $\widehat{\mathcal{N}}_{S,d}$ is the total space of the sheaf $p_*(\lambda^{\otimes 2} \otimes \pi^*\mathcal{L})$, which is coherent because p is proper. To show that $p_*(\lambda^{\otimes 2} \otimes \pi^*\mathcal{L})$ is locally free, it therefore suffices to show that its fibers have constant dimension, provided that S is reduced. Since $\deg(\lambda^{\otimes 2} \otimes \pi^*\mathcal{L}) \geq 0$ and $X = \mathbb{P}^1$,

$$h^0(X_s, \lambda_s^{\otimes 2} \otimes \mathcal{L}) = \chi(X_s, \lambda_s^{\otimes 2} \otimes \mathcal{L}) = 2d + \deg(\mathcal{L}) + 1$$

for every $s \in S$. Finally, to show that $\widehat{\mathcal{N}}_d \to \overline{\mathrm{Bun}}_{B,d}(X)$ is a vector bundle, it suffices to show that $\widehat{\mathcal{N}}_{S,d} \to S$ is a vector bundle when S is an atlas for $\overline{\mathrm{Bun}}_{B,d}(X)$. Since $\overline{\mathrm{Bun}}_{B,d}(X)$ is reduced, so is its atlas. $\square$

Due to Theorem 3.7, we will refer to $\widehat{\mathcal{N}}$ as a *global Springer resolution* when X is rational.

Corollary 3.9. *Each connected component of $\widehat{\mathcal{N}}$ has dimension $\deg(\mathcal{L}) - 1$.*

Proof. By Proposition 3.8 and Proposition 2.24, the dimension of $\widehat{\mathcal{N}}_d$ is given by

$$(2d + \deg(\mathcal{L}) + 1) + (-2d - 2) = \deg(\mathcal{L}) - 1.$$

$\square$

Corollary 3.10. *$\mathcal{N}$ is equidimensional of dimension $\deg(\mathcal{L}) - 1$.*

Proof. Since $\widehat{\mathcal{N}}$ is a resolution of singularities of $\mathcal{N}$ by Theorem 3.7, the irreducible components $\mathcal{N}_d$ of $\mathcal{N}$ are in bijection with the connected components $\widehat{\mathcal{N}}_d$ of $\widehat{\mathcal{N}}$. Furthermore, the birational equivalence between $\widehat{\mathcal{N}}_d$ and $\mathcal{N}_d$ for each d implies that their dimensions are the same. Then Corollary 3.9 implies that $\mathcal{N}$ is equidimensional of dimension $\deg(\mathcal{L}) - 1$. $\square$

3.4. **A global Springer resolution in genus 1.** X denotes an elliptic curve in this section, and $\mathcal{L}$ is allowed to be any line bundle on X of even degree at least 2. Showing that $\widehat{\mathcal{N}}$ is a resolution of $\mathcal{N}$ for an elliptic curve only requires slightly more care than it did for the projective line. Indeed, if λ is a line bundle on X of nonnegative degree, then it is almost true that $h^0(\lambda)$ is a linear function of the degree of λ. The only exception is when $\deg(\lambda) = 0$, in which case $h^0(\lambda) = 0$ or 1, depending on whether or not $\lambda = \mathcal{O}_X$.

Theorem 3.11. *When X is an elliptic curve, the projection map $\mu : \widehat{\mathcal{N}} \to \mathcal{N}$ is a resolution of singularities.*

Proof. μ is a proper, birational equivalence by Proposition 3.5 and Proposition 3.6. The smoothness of $\widehat{\mathcal{N}}_d$ for $d > -\frac{1}{2}\deg(\mathcal{L})$ follows from Proposition 3.12 and the smoothness of $\overline{\mathrm{Bun}}_B(X)$. When $d = -\frac{1}{2}\deg(\mathcal{L})$, we are only considering those $(\lambda \subset E) \in \overline{\mathrm{Bun}}_{B,d}(X)$ such that $h^0(\lambda^{\otimes 2} \otimes \mathcal{L}) \geq 1$. Then let

$$\overline{r}_{d,\mathcal{L}} : \overline{\mathrm{Bun}}_{B,d}(X) \to \mathrm{Bun}_{\mathbb{G}_m, 2d + \deg(\mathcal{L})}(X)$$

denote the map given by

$$(\lambda \subset E) \mapsto \lambda^{\otimes 2} \otimes \mathcal{L}.$$

Just as we for curves of genus at least 2, we can consider the substack $\mathcal{W}_0^0(X) \subset \mathrm{Bun}_{\mathbb{G}_m, 0}(X)$ consisting of degree 0 line bundles which possess a nonzero global section. Setting

$$\overline{\mathrm{Bun}}_{B,d}^0(X) := (\overline{r}_{d,\mathcal{L}})^{-1}(\mathcal{W}_0^0(X)),$$

the smoothness of $\widehat{\mathcal{N}}_d$ follows from Proposition 3.12, the smoothness of $\overline{r}_{d,\mathcal{L}}$, and the smoothness of $\mathcal{W}_0^0(X) \simeq B\mathbb{G}_m$. $\square$

Proposition 3.12. *First suppose that $d > -\frac{1}{2}\deg(\mathcal{L})$. Then the projection map $\widehat{\mathcal{N}}_d \to \overline{\mathrm{Bun}}_{B,d}(X)$ is a vector bundle of rank $2d + \deg(\mathcal{L})$. When $d = -\frac{1}{2}\deg(\mathcal{L})$, $\widehat{\mathcal{N}}_d \to \overline{\mathrm{Bun}}_{B,d}^0(X)$ is a line bundle.*

Proof. The proof is identical to that of Proposition 3.8. $\square$

Due to Theorem 3.11, we will refer to $\widehat{\mathcal{N}}$ as a *global Springer resolution* when X is an elliptic curve.

As in the case of genus 0, we can now extract corollaries about the equidimensionality of $\widehat{\mathcal{N}}$ and $\mathcal{N}$.

Corollary 3.13. *Each connected component of $\widehat{\mathcal{N}}$ has dimension $\deg(\mathcal{L})$. Moreover, $\mathcal{N}$ is equidimensional of dimension $\deg(\mathcal{L})$.*

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Proof. The only part of the proof not identical to Corollary 3.9 and Corollary 3.10 regards the dimension of the four connected components of $\widehat{\mathcal{N}}_{-\frac{1}{2}\deg(\mathcal{L})}$, in which case it is necessary to compute the dimension of $\overline{\mathrm{Bun}}^0_{B,-\frac{1}{2}\deg(\mathcal{L})}(X)$. Setting $d = -\frac{1}{2}\deg(\mathcal{L})$, the map

$$\overline{r}_{d,\mathcal{L}} : \overline{\mathrm{Bun}}_{B,d}(X) \to \mathrm{Bun}_{\mathbb{G}_m, 2d+\deg(\mathcal{L})}(X)$$

factors as

$$\overline{\mathrm{Bun}}_{B,d}(X) \longrightarrow \mathrm{Bun}_{\mathbb{G}_m,d}(X) \longrightarrow \mathrm{Bun}_{\mathbb{G}_m, 2d+\deg(\mathcal{L})}(X)$$

(3.2)

$$(\lambda \subset E) \longmapsto \lambda \longmapsto \lambda^{\otimes 2} \otimes \mathcal{L}$$

The second map in (3.2) is étale, and the first map is smooth by Lemma 3.14, of the same relative dimension as the corresponding projection map

$$r_d : \mathrm{Bun}_{B,d}(X) \to \mathrm{Bun}_{\mathbb{G}_m,d}(X).$$

Given a line bundle λ of degree d on X,

$$r_d^{-1}(\lambda) = \mathrm{Ext}^1(\lambda^{-1}, \lambda) / \mathrm{Hom}(\lambda^{-1}, \lambda),$$

where $\mathrm{Hom}(\lambda^{-1}, \lambda)$ acts trivially on $\mathrm{Ext}^1(\lambda^{-1}, \lambda)$. Therefore Riemann-Roch implies that the relative dimension of r_d is

$$-2d = \deg(\mathcal{L}).$$

Finally, since $\mathcal{W}_0^0(X)$ has codimension 1 in $\mathrm{Bun}_{\mathbb{G}_m,0}(X)$, we find that

$$\dim(\overline{\mathrm{Bun}}^0_{B,-\frac{1}{2}\deg(\mathcal{L})}(X)) = \deg(\mathcal{L}) - 1.$$

Finally, Proposition 3.12 implies that the corresponding connected component of $\widehat{\mathcal{N}}$ has dimension $\deg(\mathcal{L})$. $\qquad\square$

3.5. **A higher genus global Springer resolution.** In this section, X denotes a smooth, connected projective curve of genus $g \geq 2$, and $\mathcal{L}$ is a line bundle on X of even degree at least $2g$. Since the first cohomology of a line bundle on X is guaranteed to vanish if and only if the degree of the line bundle is at least $2g$, the methods used to study $\widehat{\mathcal{N}}$ when $g = 0$ or 1 will be insufficient for arbitrary g. In order to effectively analyze $\widehat{\mathcal{N}}$, we begin this section by relating the geometry of $\widehat{\mathcal{N}}$ to that of $C\mathcal{G}(X)$. The following lemma is a generalization of the well-known and easily proven fact that $\mathrm{Bun}_B(X) \to \mathrm{Bun}_T(X)$ is smooth.

Lemma 3.14. *The projection map $\overline{r} : \overline{\mathrm{Bun}}_B(X) \to \mathrm{Bun}_{\mathbb{G}_m}(X)$ is smooth.*

Proof. Since $\overline{\mathrm{Bun}}_B(X)$ and $\mathrm{Bun}_{\mathbb{G}_m}(X)$ are smooth stacks, it suffices to show that $\overline{r}$ satisfies the formal smoothness criteria for $R = \mathbb{C}[\epsilon]$ and $I = (\epsilon)$. In other words, given a $\mathbb{C}$-point $f : \lambda \hookrightarrow E$ of $\overline{\mathrm{Bun}}_B(X)$ and a first-order deformation $\phi \in H^1(X, \mathcal{O}_X)$ of λ, it suffices to find a first-order deformation $\theta \in H^1(X, \mathrm{ad}(E))$ of E and a first-order deformation $\tilde{f} : \lambda \hookrightarrow E$ of f such that the triple $(\phi, \theta, \tilde{f})$ forms a first-order deformation of $f : \lambda \hookrightarrow E$.

Let g_{ij} denote the transition 1-cocycle for λ and let h_{ij} denote the transition 1-cocycle for E, relative to a common trivializing open cover $\{U_i\}$ for λ and E. Then the condition that f is an injection from λ to E means that $f \neq 0$ and that

$$f_i g_{ij} = h_{ij} f_j,$$

along with the usual cocycle condition.

Given a fixed $\phi \in H^1(X, \mathcal{O}_X)$, a new transition 1-cocycle for the corresponding deformation of λ is given by

$$\tilde{g}_{ij} = (g_{ij} + \phi_{ij}\epsilon).$$

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Similarly, given a choice of $\theta \in H^1(X, \mathrm{ad}(E))$, a transition 1-cocycle for the corresponding deformation of E is given by

$$\tilde{h}_{ij} = (h_{ij} + \theta_{ij}\epsilon).$$

Finally, choices of $\tilde{f}_i : \lambda|_{U_i} \hookrightarrow E|_{U_i}$ give a first-order deformation of $f : \lambda \hookrightarrow E$, along with ϕ and θ, if and only if

$$(3.3) \qquad (f_i + \tilde{f}_i\epsilon)\tilde{g}_{ij} = \tilde{h}_{ij}(f_j + \tilde{f}_j\epsilon).$$

Expanding (3.3), we equivalently require that

$$(3.4) \qquad\qquad\qquad\qquad f_i g_{ij} = h_{ij} f_j,$$

$$(3.5) \qquad\qquad\qquad f_i \phi_{ij} - \theta_{ij} f_j = h_{ij}\tilde{f}_j - \tilde{f}_i g_{ij}.$$

The first equation is simply a restatement of the fact that $f : \lambda \to E$. In the second equation, the right-hand side is a coboundary, which means that $f_i\phi_{ij} - \theta_{ij}f_j = 0$ as a cohomology class. Thus it suffices to find θ such that

$$\theta f = \phi f \in H^1(X, E \otimes \lambda^{-1}).$$

Writing $f_j = \begin{pmatrix} s_j \\ t_j \end{pmatrix}$, we may then choose

$$\theta_{ij} = \begin{pmatrix} (s_j t_j + 1)\phi_{ij} & -s_j^2\phi_{ij} \\ (t_j^2 + 2\frac{t_j}{s_j})\phi_{ij} & -(s_j t_j + 1)\phi_{ij} \end{pmatrix},$$

where all occurrences of s_j and t_j denote their restrictions to U_{ij}. The fact that this choice of θ_{ij} does define a 1-cocycle with values in $\mathfrak{sl}_2$ follows from the fact that each of its entries are 1-cocycles with values in $\mathbb{C}$. $\qquad\square$

To relate $\widehat{\mathcal{N}}$ to $C\mathcal{G}(X)$, consider a variant of $C\mathcal{G}(X)$. Namely, let $C\mathcal{G}(X, \mathcal{L})$ denote the moduli stack whose S-points consist of all pairs (λ, s) where $\lambda^{\otimes 2} \otimes \pi^*\mathcal{L} \in \mathcal{W}_d^0(X)(S)$, and $s \in H^0(X_S, \lambda^{\otimes 2} \otimes \pi^*\mathcal{L})$. Then $C\mathcal{G}(X, \mathcal{L})$ fits into the following pull-back diagram.

$$(3.6)$$

$\mathrm{Sq}_{\mathcal{L}}$ is the map which sends an S-point λ to $\lambda^{\otimes 2} \otimes \pi^*\mathcal{L}$. Since the squaring map is étale and the map which tensors with a fixed line bundle is an isomorphism, the map $C\mathcal{G}(X, \mathcal{L}) \to C\mathcal{G}(X)$ is étale, which induces an étale surjection (using the fact that $\deg(\mathcal{L})$ is even)

$$C\mathcal{G}_d(X, \mathcal{L}) \to C\mathcal{G}_{2d+\deg(\mathcal{L})}(X)$$

from the degree d connected component to the degree $2d + \deg(\mathcal{L})$ connected component for each integer d. Therefore, geometric properties of $C\mathcal{G}_d(X, \mathcal{L})$ such as smoothness and irreducibility follow from the corresponding properties of $C\mathcal{G}_{2d+\deg(\mathcal{L})}(X)$. Thus we have the following corollary of Proposition 2.27.

Corollary 3.15. $C\mathcal{G}_d(X, \mathcal{L})$ *is irreducible of dimension* $2d + \deg(\mathcal{L})$. *If* $2d + \deg(\mathcal{L}) \leq 2g - 2$, *the singular locus of* $C\mathcal{G}_d(X, \mathcal{L})$ *consists of pairs* $(\lambda, 0)$ *where*

(1) $\lambda^{\otimes 2} \otimes \mathcal{L} \in \mathcal{W}_{2d+deg(\mathcal{L})}^1(X)$ *if* $2d + \deg(\mathcal{L}) < g$.

(2) $(\lambda^{\otimes 2} \otimes \mathcal{L})^{-1} \otimes \omega_X \in \mathcal{W}_{2g-2-2d-deg(\mathcal{L})}^0(X)$ *if* $g \leq 2d + \deg(\mathcal{L}) \leq 2g - 2$.

If $2d + \deg(\mathcal{L}) > 2g - 2$, *then* $C\mathcal{G}_d(X, \mathcal{L})$ *is smooth.*

To bring $\widehat{\mathcal{N}}$ into the discussion, diagram (3.6) can be extended to the following diagram in which each of the two small squares are pull-backs.

$$
\begin{array}{ccccc}
\widehat{\mathcal{N}}_d & \xrightarrow{\widetilde{r}_d} & C\mathcal{G}_d(X,\mathcal{L}) & \longrightarrow & C\mathcal{G}_{2d+\deg(\mathcal{L})}(X) \\
\downarrow & & \downarrow & & \downarrow \\
\overline{\mathrm{Bun}}_{B,d}(X) & \xrightarrow{\overline{r}_d} & \mathrm{Bun}_{\mathbb{G}_m,d}(X) & \xrightarrow{\mathrm{Sq}_{\mathcal{L}}} & \mathrm{Bun}_{\mathbb{G}_m,2d+\deg(\mathcal{L})}(X)
\end{array}
$$

The map $\widetilde{r}_d$ sends a point $(\lambda \subset E, \varphi)$ to the pair $(\lambda, \overline{\varphi})$ where $\overline{\varphi}$ is the section of $\lambda^{\otimes 2} \otimes \mathcal{L}$ corresponding to the induced map $\overline{\varphi} : \lambda^{-1} \to \lambda \otimes \mathcal{L}$.

By Lemma 3.14, it follows that $\widetilde{r}_d$ is a smooth map. Therefore questions of smoothness and irreducibility of $\widehat{\mathcal{N}}$ have been reduced to issues of smoothness and irreducibility of $C\mathcal{G}(X, \mathcal{L})$, and hence of $C\mathcal{G}(X)$. So, to completely resolve $\mathcal{N}$, it suffices to solve the simpler problem of resolving $C\mathcal{G}_d(X)$ for every $d \leq 2g - 2$.

Example 3.16. There are some special examples where $C\mathcal{G}_d(X)$ is smooth even when $d \leq 2g - 2$. The first such example occurs when $g = 3$, $d = 2$, and X is not hyperelliptic. Then Clifford's Theorem ([Har77, Thm. IV.5.4]) implies that $W_2^1(X)$ is empty, that $\mathcal{W}_2^0(X)$ is therefore smooth, and that $C\mathcal{G}_2(X)$ is a line bundle over $\mathcal{W}_2^0(X)$. More generally, if $0 \leq d \leq 2g - 2$, then $C\mathcal{G}_d(X)$ is smooth if and only if $d < g$ and $\mathcal{W}_d^0(X)$ is smooth. Recalling that $\mathcal{W}_d^0(X)$ is smooth if and only if $\mathcal{W}_d^1$ is empty, the smoothness of $C\mathcal{G}_d(X)$ depends on the existence of a g_d^1 on X. When $d \geq \frac{1}{2}g + 1$, every curve of genus g has a g_d^1, implying that $C\mathcal{G}_d(X)$ can only be smooth when $d > 2g - 2$ or $d < \frac{1}{2}g + 1$. When $d < \frac{1}{2}g + 1$, there always exists a curve of genus g which does not possess a g_d^1. See [KL74].

To resolve $C\mathcal{G}_d(X)$, consider the stack $\widetilde{C\mathcal{G}}_d(X)$, which fits into the following diagram.

$$
\begin{array}{ccccc}
\widetilde{C\mathcal{G}}_d(X) & \hookrightarrow & C\mathcal{G}_d(X) \underset{\mathcal{W}_d^0(X)}{\times} \mathcal{G}_d^0(X) & \longrightarrow & \mathcal{G}_d^0(X) \\
& \searrow^{u} & \downarrow & & \downarrow \\
& & C\mathcal{G}_d(X) & \longrightarrow & \mathcal{W}_d^0(X)
\end{array}
$$

$\widetilde{C\mathcal{G}}_d(X)$ is defined to be the closed substack of $C\mathcal{G}_d(X) \underset{\mathcal{W}_d^0(X)}{\times} \mathcal{G}_d^0(X)$ consisting of triples (λ, s, ℓ) such that $s \in \ell$.

Proposition 3.17. $\widetilde{C\mathcal{G}}_d(X) \to C\mathcal{G}_d(X)$ *is a resolution of singularities.*

Proof. The map $u : \widetilde{C\mathcal{G}}_d(X) \to C\mathcal{G}_d(X)$ is proper because $\mathcal{G}_d^0(X) \to \mathcal{W}_d^0(X)$ is proper. To show that it is a birational equivalence, consider two cases.

(1) Suppose that $d < g$. Let $z : \mathcal{W}_d^0(X) \to C\mathcal{G}_d(X)$ be the zero section, and consider the substack of $C\mathcal{G}_d(X)$ defined as the image $z(\mathcal{W}_d^1(X))$. Since $\mathcal{W}_d^1(X)$ is a closed substack of $W_d^0(X)$, $z(\mathcal{W}_d^1(X))$ is a closed substack of $C\mathcal{G}_d(X)$. Then u is a birational equivalence because it is an isomorphism on the complement of $z(\mathcal{W}_d^1(X))$.

(2) Suppose that $g \leq d \leq 2g - 2$. Then u is a birational equivalence because it is an isomorphism over the complement of the image of the zero section $z(\mathcal{W}_d^0(X))$.

$\widetilde{C\mathcal{G}}_d(X)$ is smooth because $\mathcal{G}_d^0(X)$ is smooth and the projection map $\widetilde{C\mathcal{G}}_d(X) \to \mathcal{G}_d^0(X)$ is smooth of relative dimension 1. $\qquad\square$

Remark 3.18. Unfortunately, when $g < d \leq 2g - 2$, $\widetilde{C\mathcal{G}}_d(X)$ is a somewhat unsatisfactory resolution of $C\mathcal{G}_d(X)$. This is because it is not an isomorphism over the smooth part of $C\mathcal{G}_d(X)$. Indeed, recall that there are smooth points of $C\mathcal{G}_d(X)$ lying in the image of the zero section, corresponding to those pairs $(\lambda, 0)$ such that $H^1(X, \lambda) = 0$. However, $u^{-1}((\lambda, 0)) \simeq \mathbb{P}H^0(X, \lambda)$, which is a projective space of dimension at least $d - g$. This does show, however, that u is an isomorphism over the smooth part of $C\mathcal{G}_d(X)$ when $d = g$.

For any $d \geq -\frac{1}{2}\deg(\mathcal{L})$ such that $2d + \deg(\mathcal{L}) \leq 2g - 2$, set

$$\widetilde{\mathcal{N}}_d := \widehat{\mathcal{N}}_d \underset{C\mathcal{G}_{2d+\deg(\mathcal{L})}(X)}{\times} \widetilde{C\mathcal{G}}_{2d+\deg(\mathcal{L})}(X).$$

In other words, $\widetilde{\mathcal{N}}_d$ parameterizes the data of $(\lambda \subset E, \varphi, \ell)$ such that $(\lambda \subset E, \varphi) \in \widehat{\mathcal{N}}_d$, $\ell \in \mathbb{P}H^0(X, \lambda^{\otimes 2} \otimes \mathcal{L})$, and $(\overline{\varphi} : \lambda^{-1} \to \lambda \otimes \mathcal{L}) \in \ell$.

Finally, define

$$\widetilde{\mathcal{N}} := \bigcup_{d=-\frac{1}{2}\deg(\mathcal{L})}^{g-1-\frac{\deg(\mathcal{L})}{2}} \widetilde{\mathcal{N}}_d \quad \cup \quad \bigcup_{d > g-1-\frac{\deg(\mathcal{L})}{2}} \widehat{\mathcal{N}}_d.$$

Theorem 3.19. $\widetilde{\mathcal{N}}$ *is a resolution of singularities of* $\mathcal{N}$.

$\widetilde{\mathcal{N}}$ will be referred to as a *global Springer resolution* of $\mathcal{N}$.

Corollary 3.20. *Each connected component of* $\widetilde{\mathcal{N}}$ *has dimension* $\deg(\mathcal{L}) + g - 1$. $\mathcal{N}$ *is therefore equidimensional of dimension* $\deg(\mathcal{L}) + g - 1$.

Proof. To compute the dimension of the connected component $\widetilde{\mathcal{N}}_d$, it suffices to compute the relative dimension of

$$r_d : Bun_{B,d}(X) \to Bun_{\mathbb{G}_m, d}(X).$$

Since the fiber $r_d^{-1}(\lambda)$ is the quotient stack

$$\mathrm{Ext}^1(\lambda^{-1}, \lambda) / \mathrm{Hom}(\lambda^{-1}, \lambda),$$

the relative dimension of r_d is given by the negative Euler characteristic

$$-\chi(X, \lambda^{\otimes 2}) = -2d + (g - 1).$$

Therefore,

$$\dim(\widetilde{\mathcal{N}}_d) = \dim(C\mathcal{G}_d(X, \mathcal{L})) + \dim(r_d) = \deg(\mathcal{L}) + g - 1.$$

$\square$

3.6. **Twisting bundles of smaller degree and stable bundles.** Recall the restrictions placed on the twisting bundle $\mathcal{L}$ in the previous sections. Besides requiring the degree of $\mathcal{L}$ to be even, there was also the restriction that

$$\deg(\mathcal{L}) \geq 2g.$$

The purpose of this lower bound on the degree of $\mathcal{L}$ is to ensure that for every SL_2-bundle E, there exists a nonzero nilpotent twisted endomorphism $\varphi : E \to E \otimes \mathcal{L}$. In other words, the lower bound on $\deg(\mathcal{L})$ makes it so that the image of the zero section does not form its own irreducible component of $\mathcal{N}$. To see how the particular lower bound on $\deg(\mathcal{L})$ was chosen, consider the following simple lemmas.

Lemma 3.21. *Given an* SL_2-*bundle* E, *there exists a nonzero nilpotent twisted endomorphism* $\varphi : E \to E \otimes \mathcal{L}$ *if and only if* E *possesses a line subbundle* λ *such that* $h^0(X, \lambda^{\otimes 2} \otimes \mathcal{L}) \geq 1$.

Proof. If there exists a nonzero nilpotent twisted endomorphism φ of E, then we may take $\lambda = \ker(\varphi)$.

Conversely, suppose that $\lambda \subset E$ is a line bundle such that $h^0(X, \lambda^{\otimes 2} \otimes \mathcal{L}) \geq 1$. Then pick a nonzero section $s \in H^0(X, \lambda^{\otimes 2} \otimes \mathcal{L})$, and consider $\varphi : E \to E \otimes \mathcal{L}$ defined as the composition

$$E \to E/\lambda \simeq \lambda^{-1} \xrightarrow{s} \lambda \otimes \mathcal{L} \hookrightarrow E \otimes \mathcal{L}.$$

Then φ is a nonzero nilpotent twisted endomorphism of E. $\qquad\square$

Lemma 3.22. *If λ is a line bundle on X such that*

$$\deg(\lambda) \leq -g,$$

then, for all SL_2-bundles E on X, λ is a subsheaf of E.

Proof. It is equivalent to show that $H^0(X, E \otimes \lambda^{-1}) \neq 0$. By Riemann-Roch,

$$\chi(E \otimes \lambda^{-1}) = 2(-\deg(\lambda) + 1 - g) > 0.$$

$\qquad\square$

The next proposition justifies the assumption that $\deg(\mathcal{L}) \geq 2g$.

Proposition 3.23. *If $\deg(\mathcal{L}) \geq 2g$, then any SL_2-bundle E possesses a nonzero nilpotent twisted endomorphism $\varphi : E \to E \otimes \mathcal{L}$.*

Proof. Let λ be the inverse of a square root of $\mathcal{L}$. By Lemma 3.22, λ is a subsheaf of E. If λ is furthermore a subbundle of E, then $h^0(X, \lambda^{\otimes 2} \otimes \mathcal{L}) = 1$, in which case Lemma 3.21 finishes the proof.

Otherwise, the normalization $\widetilde{\lambda} \supset \lambda$ is a subbundle of E. Then $\lambda^{\otimes 2} \otimes \mathcal{L}$ is a subsheaf of $\widetilde{\lambda}^{\otimes 2} \otimes \mathcal{L}$, which implies that

$$h^0(\widetilde{\lambda}^{\otimes 2} \otimes \mathcal{L}) \geq h^0(\lambda^{\otimes 2} \otimes \mathcal{L}) = 1.$$

Then Lemma 3.21 finishes the proof in this case as well. $\qquad\square$

Now suppose that $\deg(\mathcal{L}) \leq 2g - 2$, while keeping the assumption that $\deg(\mathcal{L})$ is even. Let Z denote the image of the zero section

$$z : \mathrm{Bun}_{SL_2}(X) \to \mathcal{N}.$$

Z is an irreducible component of $\mathcal{N}$, which is smooth because $\mathrm{Bun}_{SL_2}(X)$ is smooth. Therefore, to resolve $\mathcal{N}$ it suffices to resolve the irreducible components apart from Z. However, it is clear that $\widetilde{\mathcal{N}}$ (resp., $\widehat{\mathcal{N}}$ when $g \leq 1$) still serves this purpose, regardless of the degree of $\mathcal{L}$.

Theorem 3.24. *If $\mathcal{L}$ is an even degree line bundle such that $\deg(\mathcal{L}) \leq 2g - 2$, then the disjoint union $\widetilde{\mathcal{N}} \sqcup Z$ (resp., $\widehat{\mathcal{N}} \sqcup Z$) is a resolution of singularities of $\mathcal{N}$.*

Having now discussed $\mathcal{N}$ for any even degree line bundle $\mathcal{L}$, let us briefly turn our attention to stable Higgs bundles. We assume that $g \geq 2$ for the remainder of the section, as there are no stable SL_2-bundles when $g < 2$. The following definition first appeared in [Hit87a].

Definition 3.25. For any vector bundle E, the *slope* of E is defined to be

$$\mu(E) := \frac{\deg(E)}{\mathrm{rk}(E)}.$$

A Higgs bundle (E, φ) is said to be *stable* if, for any vector bundle F which is a φ-invariant subsheaf of E,

$$\mu(F) < \mu(E).$$

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Let $\mathcal{N}^s \subset \mathcal{N}$ denote the substack of stable nilpotent Higgs bundles. Note that if (E, φ) is a stable nilpotent Higgs bundle, then $F \subset E$ is a φ-invariant subsheaf if and only if $F \subset \ker(\varphi)$. Let $\widetilde{\mathcal{N}}_d^s$ denote the restriction of $\widetilde{\mathcal{N}}_d$ to $\mathcal{N}_d^s$, and set

$$\widetilde{\mathcal{N}}^s = \bigcup_{d=-1}^{-\frac{1}{2}\deg(\mathcal{L})} \widetilde{\mathcal{N}}_d^s.$$

Proposition 3.26. *First suppose that* $\deg(\mathcal{L}) \geq 2g$. *Then* $\widetilde{\mathcal{N}}^s$ *is a resolution of singularities of* $\mathcal{N}^s$. *If* $0 < \deg(\mathcal{L}) \leq 2g - 2$, *then* $\widetilde{\mathcal{N}}^s \sqcup Z$ *is a resolution of singularities of* $\mathcal{N}^s$. *Lastly, if* $\deg(\mathcal{L}) \leq 0$, *then* $\mathcal{N}^s = Z$ *is smooth.*

In particular, Proposition 3.26 provides a count of the finite number of irreducible components of $\mathcal{N}^s$.

Corollary 3.27.   (1) *If* $\deg(\mathcal{L}) \geq 2g$, *then* $\mathcal{N}^s$ *has* $\frac{1}{2}\deg(\mathcal{L})$ *irreducible components.*
    (2) *If* $0 < \deg(\mathcal{L}) \leq 2g - 2$, *then* $\mathcal{N}^s$ *has* $\frac{1}{2}\deg(\mathcal{L}) + 1$ *irreducible components.*
    (3) *If* $\deg(\mathcal{L}) \leq 0$, *then* $\mathcal{N}^s$ *has* 1 *irreducible component.*

References

[ACGH85] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris, *Geometry of algebraic curves. Vol. I*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 267, Springer-Verlag, New York, 1985. MR 770932 (86h:14019)

[AK70] Allen Altman and Steven Kleiman, *Introduction to Grothendieck duality theory*, Lecture Notes in Mathematics, Vol. 146, Springer-Verlag, Berlin, 1970. MR 0274461 (43 #224)

[BD] Alexander Beilinson and Vladimir Drinfeld, *Quantization of Hitchin's integrable system and Hecke eigensheaves*, available at `http://www.math.uchicago.edu/ mitya/langlands/hitchin/BD-hitchin.pdf`.

[BG02] A. Braverman and D. Gaitsgory, *Geometric Eisenstein series*, Invent. Math. **150** (2002), no. 2, 287–384. MR 1933587 (2003k:11109)

[BM83] Walter Borho and Robert MacPherson, *Partial resolutions of nilpotent varieties*, Analysis and topology on singular spaces, II, III (Luminy, 1981), Astérisque, vol. 101, Soc. Math. France, Paris, 1983, pp. 23–74. MR 737927 (85j:14087)

[Bou68] N. Bourbaki, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*, Actualités Scientifiques et Industrielles, No. 1337, Hermann, Paris, 1968. MR 0240238 (39 #1590)

[Bra06] Alexander Braverman, *Spaces of quasi-maps into the flag varieties and their applications*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 1145–1170. MR 2275639 (2008i:14019)

[CG97] Neil Chriss and Victor Ginzburg, *Representation theory and complex geometry*, Birkhäuser Boston Inc., Boston, MA, 1997. MR 1433132 (98i:22021)

[CL10] Pierre-Henri Chaudouard and Gérard Laumon, *Le lemme fondamental pondéré. I. Constructions géométriques*, Compos. Math. **146** (2010), no. 6, 1416–1506. MR 2735371

[dCHM12] Mark Andrea de Cataldo, Tamas Hausel, and Luca Migliorini, *Topology of Hitchin systems and Hodge theory of character varieties: the case of A_1*, to appear in Ann. of Math., 2012.

[Eis95] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry. MR 1322960 (97a:13001)

[Fal93] Gerd Faltings, *Stable G-bundles and projective connections*, J. Algebraic Geom. **2** (1993), no. 3, 507–568. MR 1211997 (94i:14015)

[FM99] Michael Finkelberg and Ivan Mirković, *Semi-infinite flags. I. Case of global curve $\mathbf{P}^1$*, Differential topology, infinite-dimensional Lie algebras, and applications, Amer. Math. Soc. Transl. Ser. 2, vol. 194, Amer. Math. Soc., Providence, RI, 1999, pp. 81–112. MR 1729360 (2001j:14029)

[Fre07] Edward Frenkel, *Lectures on the Langlands program and conformal field theory*, Frontiers in number theory, physics, and geometry. II, Springer, Berlin, 2007, pp. 387–533. MR 2290768 (2007k:11102)

[Gin01] Victor Ginzburg, *The global nilpotent variety is Lagrangian*, Duke Math. J. **109** (2001), no. 3, 511–519. MR 1853354 (2002j:14016)

[Gri98]   Mikhail Grinberg, *A generalization of Springer theory using nearby cycles*, Represent. Theory **2** (1998), 410–431 (electronic). MR 1657203 (2000a:20099)

[Har77]   Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157 (57 #3116)

[Hei]   Jochen Heinloth, *Lecture on the moduli stack of vector bundles on a curve*, available at `http://staff.science.uva.nl/ heinloth/`.

[Hit87a]   N. J. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. (3) **55** (1987), no. 1, 59–126. MR 887284 (89a:32021)

[Hit87b]   Nigel Hitchin, *Stable bundles and integrable systems*, Duke Math. J. **54** (1987), no. 1, 91–114. MR 885778 (88i:58068)

[Hot81]   Ryoshi Hotta, *On Springer's representations*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), no. 3, 863–876 (1982). MR 656061 (83h:20038)

[HTT08]   Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki, *D-modules, perverse sheaves, and representation theory*, Progress in Mathematics, vol. 236, Birkhäuser Boston Inc., Boston, MA, 2008, Translated from the 1995 Japanese edition by Takeuchi. MR 2357361 (2008k:32022)

[Ill71]   Luc Illusie, *Complexe cotangent et déformations. I*, Lecture Notes in Mathematics, Vol. 239, Springer-Verlag, Berlin, 1971. MR 0491680 (58 #10886a)

[KL74]   Steven L. Kleiman and Dan Laksov, *Another proof of the existence of special divisors*, Acta Math. **132** (1974), 163–176. MR 0357398 (50 #9866)

[KM85]   Nicholas M. Katz and Barry Mazur, *Arithmetic moduli of elliptic curves*, Annals of Mathematics Studies, vol. 108, Princeton University Press, Princeton, NJ, 1985. MR 772569 (86i:11024)

[Kuz97]   Alexander Kuznetsov, *Laumon's resolution of Drinfeld's compactification is small*, Math. Res. Lett. **4** (1997), no. 2-3, 349–364. MR 1453065 (98d:14022)

[Lau87]   Gérard Laumon, *Correspondance de Langlands géométrique pour les corps de fonctions*, Duke Math. J. **54** (1987), no. 2, 309–359. MR 899400 (88g:11086)

[Lau88]   ⸺, *Un analogue global du cône nilpotent*, Duke Math. J. **57** (1988), no. 2, 647–671. MR 962524 (90a:14012)

[Lus81]   G. Lusztig, *Green polynomials and singularities of unipotent classes*, Adv. in Math. **42** (1981), no. 2, 169–178. MR 641425 (83c:20059)

[Lus96]   George Lusztig, *Affine Weyl groups and conjugacy classes in Weyl groups*, Transform. Groups **1** (1996), no. 1-2, 83–97. MR 1390751 (97g:20045)

[MW84]   B. Mazur and A. Wiles, *Class fields of abelian extensions of* **Q**, Invent. Math. **76** (1984), no. 2, 179–330. MR 742853 (85m:11069)

[Nad12]   David Nadler, *The geometric nature of the fundamental lemma*, Bull. Amer. Math. Soc. **49** (2012), 1–50.

[Ngô10]   Bao Châu Ngô, *Le lemme fondamental pour les algèbres de Lie*, Publ. Math. Inst. Hautes Études Sci. (2010), no. 111, 1–169. MR 2653248

[Sha72]   J. A. Shalika, *A theorem on semi-simple P-adic groups*, Ann. of Math. (2) **95** (1972), 226–242. MR 0323957 (48 #2310)

[Sim92]   Carlos T. Simpson, *Higgs bundles and local systems*, Inst. Hautes Études Sci. Publ. Math. (1992), no. 75, 5–95. MR 1179076 (94d:32027)

[Slo80]   Peter Slodowy, *Simple singularities and simple algebraic groups*, Lecture Notes in Mathematics, vol. 815, Springer, Berlin, 1980. MR 584445 (82g:14037)

[Spr76]   T. A. Springer, *Trigonometric sums, Green functions of finite groups and representations of Weyl groups*, Invent. Math. **36** (1976), 173–207. MR 0442103 (56 #491)

[Yun11]   Zhiwei Yun, *Global Springer theory*, Adv. Math. **228** (2011), no. 1, 266–328. MR 2822234